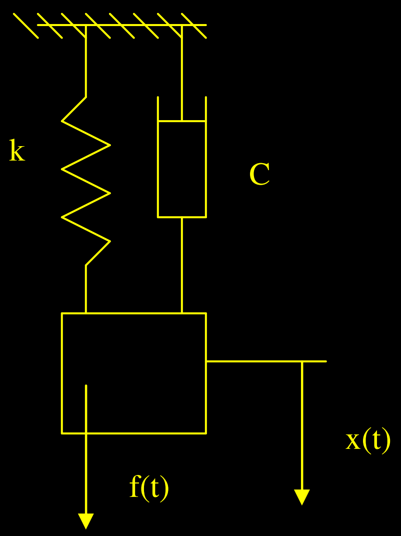
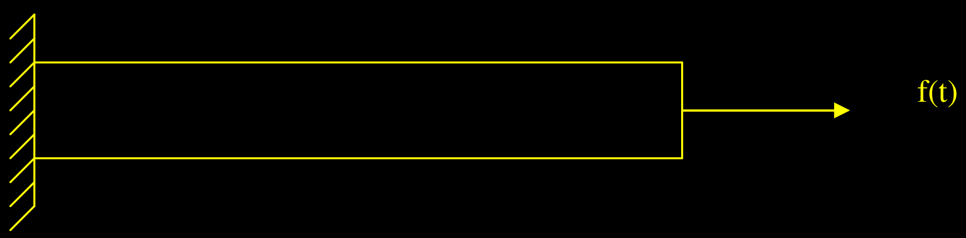
Dynamique des structures

1° Système à masse et élasticité dissociées

$\omega = \sqrt{k/m}$

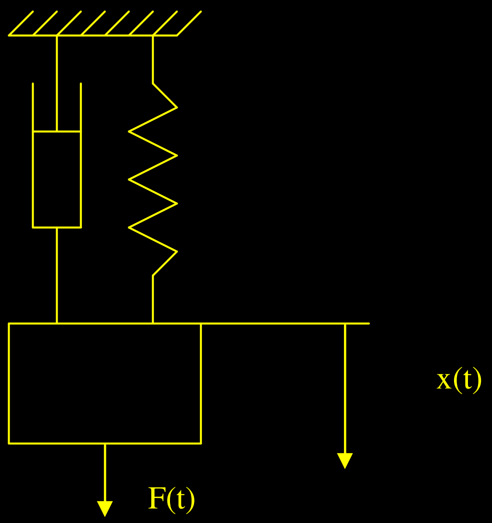


2° Système à masse et élasticité associées



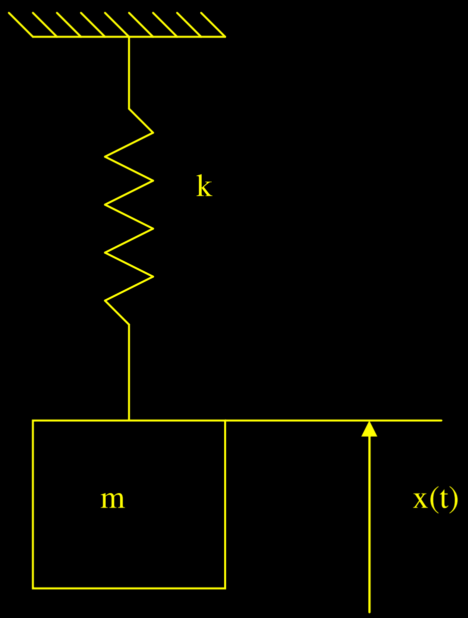
I Système à masse et élasticité dissocié

1° Système à un d° de liberté



$$\Sigma F/x = \begin{vmatrix} -kx - cx' = mx'' \\ F(t) \end{vmatrix}$$

- Système libre non amorti



$$Mx'' + kx = 0$$
$$x = a \cos \omega n\, t + b \sin \omega n\, t$$

Où $\omega_n = \sqrt{k/m}$ est la pulsation propre (rd/s)
a et b dépendent des conditions initiales

Ex :

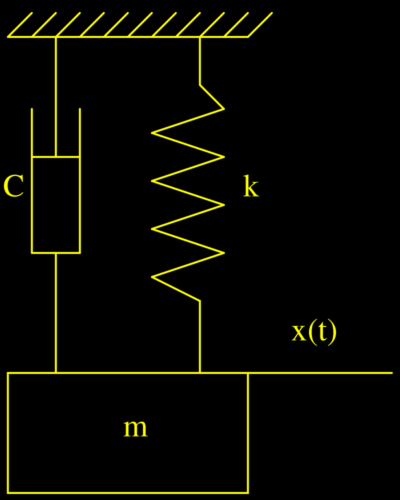
$x(0) = x_0$

$x'(0) = 0 \Rightarrow b=0$

$x = x_0 \cos \omega_n t$

période $T = 2\Pi/\omega$ (s) fréquence : $f=\omega/2\Pi$

- Système libre amorti



- $mx'' + Cx' + kx = 0$
- $x'' + C/m\, x' + \omega^2 n\, x = 0$

$C/m = 2\, \alpha\omega_n$ avec α facteur d'amortissement

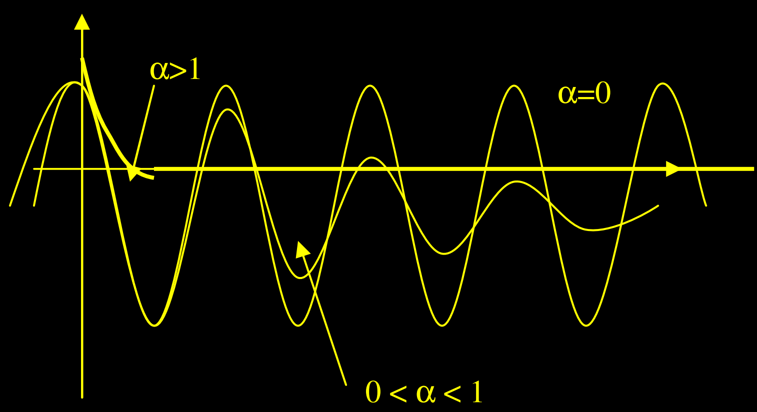
$x = Pr[Ze^{rt}]$ avec $r^2 = -\omega^2$ pour $\alpha<1 \Leftrightarrow$ r complexe

- $r^2 + 2\alpha\omega_n r + \omega^2_n = 0$

Pour $\alpha<1$ $r1{,}2 = -\alpha\omega_n \pm i\, \omega_n \sqrt{1-\alpha}$

$x = Pr(Z_1 e^{r1t} + Z_2 e^{r2t})$

$x = A\, e^{-\alpha\omega nt} \sin(\sqrt{1-\alpha^2}\ \ \omega_n t + \psi)$



- <u>Système forcé :</u>

$m x''(t) + C x' + hx = F(t)$
$x(t) = A e^{(-\alpha\omega n t)} \sin(\omega t + \psi) + \text{sol particulière } (F(t))$
$x(t) = \text{sol particulière } F(t)$ pour $t = \infty$

Ex :

$F(t) = F \sin \Omega t$ moteur avec balourd

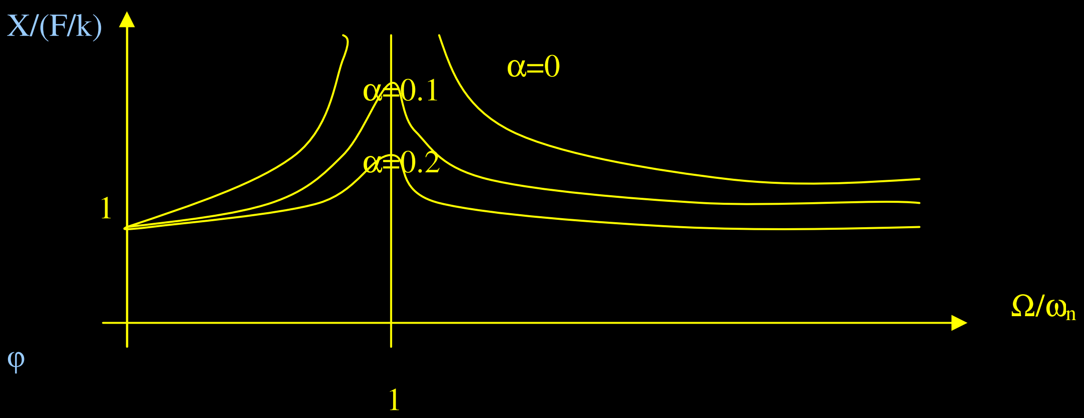
<u>Solution :</u>

$x(t) = x \sin(\Omega t - \varphi)$
$x'(t) = x\,\Omega \cos(\Omega t - \varphi)$ $x''(t) = -x\,\Omega \sin(\Omega t - \varphi)$
x : Amplitude
φ : Déphasage $\omega n = \sqrt{k/n}$

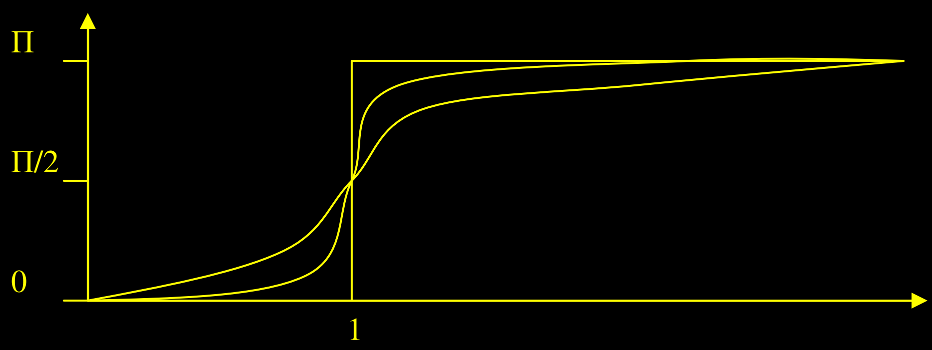
<u>Sol :</u>

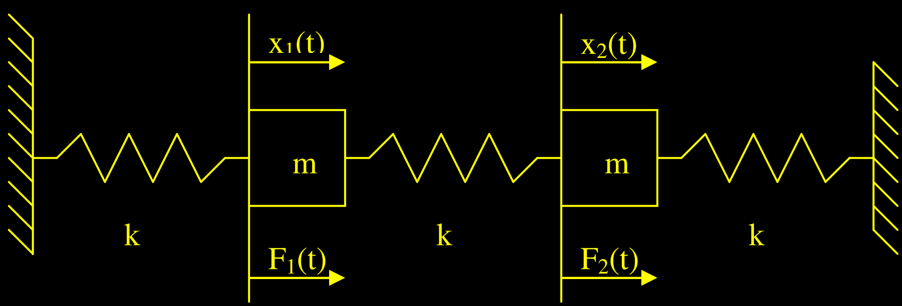
$$\mathrm{tg}\,\varphi = (2\alpha\,\Omega/\omega_n)/(1-(\Omega/\omega_n)^2) = 2\alpha\omega_n\Omega/(\omega_n^2 - \Omega^2)$$

$X/(F/k) = 1/\sqrt{[1-(\Omega/\omega_n)^2]^2 + (2\alpha\Omega/\omega_n)^2}$
$X/(F/k) = 1/\sqrt{(\omega^2_n - \Omega^2)^2 + (1\alpha\Omega\omega_n)^2)}$



φ





Principe Fondamental de la Dynamique

Masse 1

$F_1(t) - kx_1 - k(x_1 - x_2) = m\, x''_1$

Masse 2

$F_2(t) – k(x_2 – x_1) – kx_2 = mx''_2$

2 Système à 2 d° de liberté

2.1. Système non amorti

$Mx''_1 + 2kx_1 – kx_2 = F_1(t)$
$Mx''_2 – kx_1 + 2kx_2 = F_2(t)$

Remarque :

Sous forme matricielle :

$M\, x'' + K\, x = F$

$$\underset{\text{Matrice Masse}}{\begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix}} \cdot \underset{\text{Vecteur Déplacement}}{\begin{vmatrix} x''_1 \\ x''_2 \end{vmatrix}} + \underset{\text{Matrice de Raideur}}{\begin{vmatrix} 2k & -k \\ -k & -2k \end{vmatrix}} \cdot \underset{\text{Vecteur Déplacement}}{\begin{vmatrix} x_1 \\ x_2 \end{vmatrix}} = \underset{\text{Forces Généralisée}}{\begin{vmatrix} F_1(t) \\ F_2(t) \end{vmatrix}}$$

- Mouvement libre ($F(t) = 0$)

$X = X\, e^{rt}$ ou $x_1 = X_1\, e^{rt}$
$x_2 = X_2\, e^{rt}$

ou $x_1 = A_1 \cos \omega n\, t + B_1 \sin \omega n\, t$
$x_2 = A_2 \cos \omega n\, t + B_2 \sin \omega n\, t$

$(mr^2 + 2k)\, x_1 – kx_2 = 0$

- $kx1 + mr^2 + 2kX2 = 0$

$(Mr^2+K)X = 0$
$X=0$ ou $\det (Mr^2+K) = 0$

$$\text{Det} \begin{vmatrix} mr^2 + 2k & -k \\ -k & mr^2 + 2k \end{vmatrix} = 0$$

$$(mr^2 + 2k)^2 - k^2 = (mr^2 + k)(mr^2 + 3k) = 0$$

- $r_1^2 = -k/m = -\omega_1^2$ et $r_2^2 = -3k/m = -\omega_2^2$
  $r1 = \pm i(\omega_1)$ $r2 = \pm i(\omega_2)$

- pour $r_1^2 = -k/m$

$kX_1 - kX_2 = 0 \quad \Rightarrow X_1 = X_2$

- $kX_1 + kX_2 = 0$

Vecteur normé :

$$X \begin{vmatrix} (\sqrt{2})/2 \\ (\sqrt{2})/2 \end{vmatrix}$$

pour $r_2^2 = -3k/m$

⇨ $-kX_1 - kX_2 = 0 \quad \Rightarrow \quad X_1 = -X_2$
$kX_1 - KX_2 = 0$

Vecteur normé :

$$X \begin{vmatrix} (\sqrt{2})/2 \\ (-\sqrt{2})/2 \end{vmatrix}$$

Solution générale

$$x_1(t) = \alpha_1^{ei\omega 1t} + \beta_1 e^{-i\omega 1t} + \alpha_2^{ei\omega 2t} + \beta_2 e^{-i\omega 2t}$$
$$x_2(t) = \alpha_1^{ei\omega 1t} + \beta_1 e^{-i\omega 1t} - \alpha_2^{ei\omega 2t} - \beta_2 e^{i\omega 2t}$$

$$x_1 = a_1\cos\omega_1 t + b_1\sin\omega_1 t + a_2\cos\omega_2 t + b_2\sin\omega_2 t$$
$$x_2 = a_1\cos\omega_1 t + b_1\sin\omega_1 t - a_2\cos\omega_2 t - b_2\sin\omega_2 t$$

Théorème de l'énergie

Methode des éléments finis
Energie potentielle

$Ep(U_s) < Ep(U)\ \forall\ U$ CA avec $U = Ud$ sur $\partial_1 \Omega$

Résistance des matériaux

Energie complémentaire

$Ec(\sigma_s) < Ec(\sigma)\ \forall\ \sigma$ SA avec div $\sigma + fd = 0\ \Omega$ et $\sigma n = Fd$ sur $\partial_2\ \Omega$

Dem

$E_p\ \ U = Us + U_0$

$U = U_d$ $\quad$ $U_s = U_d$ $\quad$ $U_0 = 0$
$\partial_1\Omega$ $\quad$ $\partial_1\Omega$ $\quad$ $\partial_1\Omega$

$E_p(u) = ½\ {}_{\Omega}\int Tr[\varepsilon(U_s + U_0)\ K\ \varepsilon\ (U_s + U_0)]\ d\Omega - {}_{\Omega}\int f_d\ (U_s + U_0) d\Omega - {}_{\partial 2\Omega}\int F_d(U_s+U_0)dS$
$= E_p(U_s) + ½\ {}_{\Omega}\int Tr[\varepsilon(U_0)K\varepsilon(V_0)]d\Omega + {}_{\Omega}\int Tr\ [\varepsilon(u_0)K\varepsilon(U_s)]d\Omega - {}_{\Omega}\int f_d U_0 d\Omega - {}_{\partial 2\Omega}\int F_d U_o dS$

$U_0 = U^*$ => TPV

Approximation

$U_n = \Sigma\ a_i\ u_i$

Les ai minimisent $E_p(U_n)$ =>U_n : C.A.

Dem :

E_c

$\sigma = \sigma_s + \sigma_0$

$|\text{div}\ \sigma + f_d = 0$ $\quad$ $|\text{div}\sigma_s + f_d = 0$ $\quad$ $|\text{div}\sigma_0 = 0$ dans Ω
$|\sigma n = F_d$ $\quad$ $|\ \sigma_s n = Fd$ $\quad$ $|\sigma_0 n = 0$
$|\ \ \partial_2\Omega$ $\quad$ $|\ \ \partial_2\Omega$ $\quad$ $|\ \ \partial_2\Omega$

$E_c\ (\sigma = \sigma_s + \sigma_0) = 1/2\ \ {}_{\Omega}\int Tr[(\sigma_s + \sigma_0)\ K^{-1}\ (\sigma_s + \sigma_0)]\ d\Omega$

$E_c(\sigma = \sigma_s + \sigma_0)\ = 1/2\ {}_{\Omega}\int Tr[(\sigma_s + \sigma_0)\ K^{-1}\ (\sigma_s + \sigma_0]\ d\Omega - {}_{\partial 1\Omega}\int(\sigma_s + \sigma_0)\ n\ U_d\ dS$
$= E_c\ (\sigma_s) + ½\ {}_{\Omega}\int\ Tr\ [\sigma_0 K^{-1} \sigma_0]d\Omega + {}_{\Omega}\int\ Tr\ [\sigma_0\ K^{-1}\ \sigma_s]d\Omega - {}_{\partial 1\Omega}\int\sigma_0\ n\ U_d\ dS$

(fg)' –f'g = fg'

$fg' = {}_{\Omega}\int\ \text{div}\ (\sigma_0\ U_s)\ d\Omega - {}_{\Omega}\int\ \text{div}\ \sigma_0\ U_s\ d\Omega$

${}_{\partial\Omega}\int\ \sigma_0\ n\ U_s\ dS = {}_{\partial 1}\int\ \sigma_0\ n\ U_s\ dS + \int\sigma_0\ n\ U_s\ dS$

${}_{\Omega}\int\ \varphi_{,i}\ d\Omega = {}_{\partial\Omega}\int\ \varphi n_i\ dS$

$Tr[\sigma\ \varepsilon\ ]$ $\quad$ $\sigma : \varepsilon = \sigma_{ij}\ \varepsilon_{ij}$
$u\ .\ v = u_i.v_i$

Calcul préliminaire (1^{er} dimensionnement)

1° Calcul de EC

$$E_c(\sigma) = E_c(\sigma_s) + ½ \int_\Omega Tr [\sigma_0 K^{-1} \sigma_0] d\Omega$$

2° Verification de σ

$$\sigma \text{ SA}$$

3° Minimum Ec par σ

$$E_c(\sigma) > E_c(\sigma_s) \Leftarrow E_d(\sigma_0) > 0$$
$$\forall \sigma \text{ S A}$$

Calcul des répartitions

Approximations

$$\sigma_n = \sum_{i=1}^{a} a_i \sigma_i (x,y,z) \; \sigma_n \text{ SA}$$

μ.
.!

Les a_i minimisent $E_c(\sigma_n)$

Ex Rdm :

R(X,Y,M) vérifient l'équilibre
M(X,Y,M)

$\partial E_c(x,y)/\partial x = 0$
$\partial E_c/\partial y = 0$

4° Propriétés d'encadrement

$E_p(U) > E_p(U_s) = - E_p(\sigma_s) >- E_c(\sigma)$

Théorème des travaux virtuels avec $U^* = U_s$; σ_s

- $-\int_\Omega Tr [\sigma_s \varepsilon(U_s)] d\Omega + \int_\Omega f_d U_s d\Omega + \int_{\partial 2\Omega} F_d U_s dS + \int_{\partial 1\Omega} \sigma_s n U_s dS = 0$

- $$\begin{aligned}(½ + ½) \int_\Omega Tr [\sigma_s \varepsilon(U_s)] d\Omega = & \; ½ \int_\Omega Tr[K \varepsilon(U_s) \varepsilon(U_s)] d\Omega \\ & + ½ \int_\Omega tr [\sigma_S K + \sigma_s] d\Omega \\ & - ½ \int_\Omega Tr [\varepsilon(U_s) K \varepsilon(U_s)] d\Omega \\ & + ½ \int_\Omega f_d U_s d\Omega \\ & + ½ \int_{\partial 2\Omega} F_d U_s dS \\ & - ½ \int_\Omega Tr [\sigma_s K^{-1} \sigma_s] d\Omega \\ & + \int_{\partial 1\Omega} \sigma_s n U_d dS \\ = & \; 0\end{aligned}$$

- $E_p\ (U_s) - E_c\ (\sigma_s) = 0$

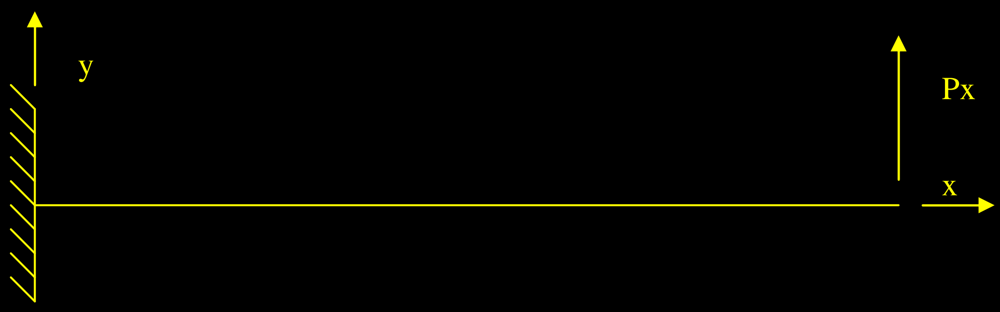
Application:

Encadrement des propriétés mécaniques des matériaux composites

5° Application du théorème de l'énergie potentielle (poutre en flexion)

Hypothèse de bernouilli

On cherche le déplacement suivant : y : v(x)



$V(0) = 0$
$\omega(0) = d\omega/dx\ (0) \neq 0$

Théorème de l'énergie potentielle Vn(x) C.A.

$$V_n = \sum_{i=0}^{n} a_i\, x^i$$

$$= a_0\, x^0 + a_1\, x + a_2 x^2 + \ldots$$

V_n CA =>
$V_n\ (0) = 0 \Rightarrow a_0 = 0$
$D\ V_n\ /\ dx\ (0) = 0 \Rightarrow a_1 = 0$

1 ière Approximation $V_2 = a_2 x^2$

$E_p\ (V_2)$ a_2 minimise $E_p\ (a_2)$

V_1 CA $\Leftrightarrow$ $\partial E_p\ /\ \partial a_2 = 0$

$E_p(v) = \frac{1}{2} \int_0^L EI\ (v'')^2\ dx - F_d\ v(L)$
$E_d(v)$

$V_2 = 2a_2$

$E_p(v) = ½\, EI\, 4\, a_2^2\, L - F_d\, a_2\, L^2$

$\partial E_p / \partial a_2 = 4\, EIL\, a_2 - F_d\, L^2 = 0 \quad \Rightarrow a_2 = F_d L / 4EI$

2 ieme approximation $V_3(x) = a_2 x^2 + a_3\, x^3$ C. A.

$V''_3 = 2a_2 + 6\, a_3\, x$

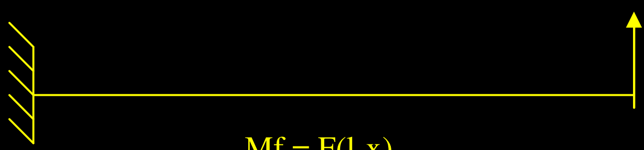
$E_p(v) = ½\, EI\, {}_0\int^L (2a_2 + 6a_1 x)^2\, dx - F_d\, (a_2\, L^2 + a_3\, L^3)$

$\partial E_p/\partial a_2 = 0 = EI\, {}_0\int^L 2(2a_2 + 6a_3 x)\, dx - F_d L^2 = 0$

$\partial E_p/\partial a_3 = 0 = EI\, {}_0\int^L 6x(2a_2 + 6a_3 x)\, dx - F_d L^3 = 0$

$EI\, (\, 4\, a_2\, L + 12\, a_3\, L^2/2) = F_d L^2$
$EI\ \ 2 * 6a_2\, L^2/2 + 6 * 6\, a_3\, L^3\, /3\,) = F_d L^3$

$a_2 = F_d l\, /\, 2EI \quad a_3 = -F_d\, /\, 6EI$
$a_2\, x_2^2\ + a_3\, x_3^2 + a_4\, x_4^2$



Mf = F(l-x)

$EI\, v'' = Mf = F(L-x)$
$EI\, v' = F(Lx - x^2/2)$
$EI\, v = F\, (Lx^2/2 - x^3/6)$

Methode analytique

En L $y(L) = 0 \Leftrightarrow Y_AL^3/6EI - Pl^3/6EI + PlL^2/2EI + K_3L + K_4$
En l $y'(l+) = y'(l-) \Leftrightarrow Y_Al^2/2EI + K_1 - Y_1l^2/2EI + Pl^2/2EI - Pl^2/EI - K_3 = 0$
$y(l+) = y(l-) \Leftrightarrow Y_Al^3/6EI - Pl^3/6EI + Pll^2/2EI + K_3l + K_4$

1ere Intégrale

$y' = Y_AS^2/2EI + K_1$
$y' = -Y_AS^2/2EI - PS^2/2EI + Pls/EI + K_3$

2 iéme Intégrale

$y = Y_AS^3/6EI + K_1S + K_2$
$y = -Y_AS^3/6EI - Ps^3/6EI + Pls^2/2EI + K_3s + K_4$
$y'(l) = -Pl^2/2EI$
$y(l) = Y_Al^3/6EI + Pl^3/3EI = (Y_A/2 + P)\, l^3/EI$

T => y''
Mf =>y'
U => y

1 0 -sinα
0 1 cosα
0 0 lcosα

P = [0 P lP]
C=sin[]
…..
Methode analytique
LOn g !

Chapitre III - Introduction à la methode des éléments Finis (MEF) ang (FEN)

I) Introduction

a) Principe

Approximation :

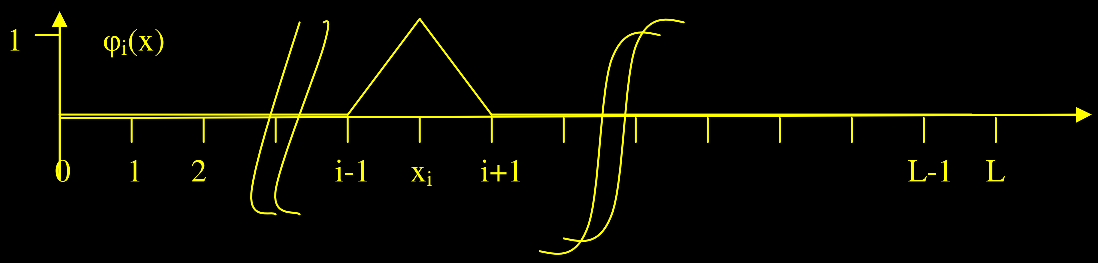
1) Découpage en éléments finis
2) Choix d'un champ de déplacement
3) La meilleure approximation minimise E_p (App.)

$$U_n = \Sigma a_i \varphi_i(x,y,z)$$
$$\text{Min } E_p (a_i) \Rightarrow \partial E_p / \partial a_1 = 0$$
$$U_n = CA$$

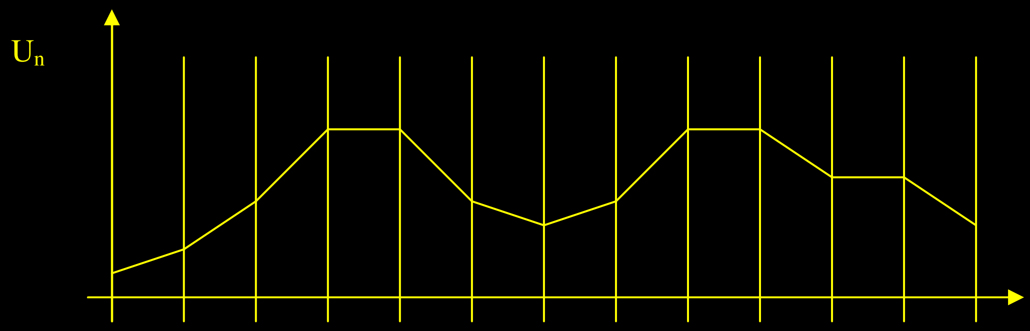
Méthode de Ritz Galerkin:

φ_i très régulière et définie sur tout Ω (polynômes en 1D x, x^2, x^3...)
Methode des éléments finis :
φ_i peu régulière et définie sur une partie de Ω

ex : ID



$$U_n(x) = \sum_{i=1}^{n} U_i \varphi_i(x)$$



Propriété

$U_n\ (x_i) = U_1\varphi_1(x_i) + U_2\varphi_2(x_i) + \ldots + U_{i-1}\varphi_{i-1}(x_i) + U_i\varphi_i(x_i) + U_{i+1}\varphi_{i+1}(x_i) + \ldots + U_n\varphi_n\ (x_i)$

$U_n\ (x_i) = U_i$ Coeff de la fonction φ_i

b) Formulation matricielle

avec:

$$U_n = \sum_{i=1}^{n} U_i\,\varphi_i\ (x,y,z)$$

$E_p(U_n)$

U_n CA

$U_n = U_d$ sur $\partial_1\Omega$

$U_n = U_d$ sur $\partial_1\Omega$

U(0) = 0 Un = 0

Th Ep élét fini

Th Ec poutres

$$E_p\ (U_n) = \underbrace{1/2\ {}_{\Omega}\!\int \mathrm{Tr}\ [\varepsilon\ (\Sigma U_i)\ \varphi_i\ K\varepsilon(\Sigma U_j\varphi_j)]\ d\Omega}_{E_d\ (U_n)}\ -\ \underbrace{{}_{\Omega}\!\int f_d\ (\ \Sigma\ u_i\ \varphi_i\ )\ d\Omega - {}_{\Omega}\!\int F_d\ (\ \Sigma\ U_i\ \varphi_i)\ dS}_{\text{Travail des efforts donnés dans } U_n}$$

ex :

$$U_n = U_1\varphi_1 + U_2\varphi_2$$

$$\varepsilon\ (U_n) = U_1\ \varepsilon\ (\varphi_1) + U_2\ \varepsilon\ (\varphi_2)$$

$$Ep\ (U_2) = 1/2 \sum_{i=1}^{n}\sum_{j=1}^{n} u_i\ u_j\ {}_{\Omega}\!\int \mathrm{Tr}\ [\ \varepsilon\ (\ \varphi_i)\ K\ \varepsilon\ (\varphi_i)\ ]\ d\Omega - \Sigma\ u_i\ [\ {}_{\Omega}\!\int f_d\ \varphi_i\ d\Omega + \partial_2\Omega\!\int F_d\ \varphi_i dS]$$

ex :

$$1/2[\ u_1^2\ {}_{\Omega}\!\int \mathrm{Tr}[\varepsilon(\varphi_1^2)K\varepsilon(\varphi_1^2)]d\Omega + U_2^2\,{}_{\Omega}\!\int \mathrm{Tr}[\varepsilon(\varphi_2)K\varepsilon(\varphi_2)]d\Omega + 2U_1U_2\ {}_{\Omega}\!\int \mathrm{Tr}(\varepsilon(\varphi_1)K\varepsilon(\varphi_2)d\Omega]$$

$$E_p = 1/2 \sum_{i=1}^{n} \sum_{j=1}^{n} u_i\, u_j\; K_{ij} \,.\, \sum u_i\, F_i$$

où

$$K_{ij} = {}_{\Omega}\!\int Tr\, [\varepsilon(\varphi_i) K \varepsilon(\varphi_j)]\; d\Omega$$

K matrice de rigidité dim (nxn)

et

$$F_i = {}_{\Omega}\!\int f_d\, \varphi_i\, d\Omega + {}_{\partial 2\Omega}\!\int F_d\, \varphi_i\, dS$$

F Forces généralisées dim (n)

On définit également :

$$U : \begin{matrix} U_1 \\ U_2 \\ U_3 \\ : \\ : \\ U_n \end{matrix}$$

$$E_p = 1/2\; U^T\, K\, U - U^T\, F$$

Ex:

$$U_n = U_1\, \varphi_1 + U_2\, \varphi_2$$

$$E_p = 1/2\; (U_1^2 K_{11} + U_2^2 K_{12} + U_1 U_2 K_{12} + U_2 U_1 K_{21})$$

$$U \quad \begin{matrix} U_1 \\ U_2 \end{matrix}$$

$$|F_1| \quad = 1/2\; [U_1,U_2] \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \begin{vmatrix} U_1 \\ U_2 \end{vmatrix} - [U_1,U_2] |F_2|$$

$$E_p = 1/2\; U^T\, KU - U^T\, F$$

$$E_p(U'=(U_1,U_2,\ldots\ldots,U_n)$$

$$\text{Min } E_p(U_1,U_2,\ldots\ldots,U_n) \Leftrightarrow \partial E_p/\partial u_i = 0$$

ou

$$\partial E_p/\partial U_1 = 0\; ;\; \partial E_p/\partial U_2 = 0\; ;\; \ldots\ldots\ldots$$

$$\Leftrightarrow K\, U - F = 0$$

=> $\partial E_p/\partial u_1 = 0 = K_{11}U_1 + K_{12}\,U_2 - F_1 = 0$
$\partial E_p/\partial u_2 = 0 = K_{22}U_2 + K_{21}U_1 - F_2 = 0$

$$\begin{vmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{vmatrix} \begin{vmatrix} U_1 \\ U_2 \end{vmatrix} \quad \begin{vmatrix} F_1 \\ -F_2 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$K\;U\;-F = 0$
$K\,U = F$

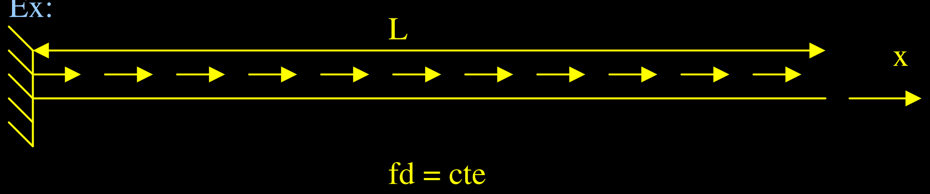
<u>Remarque :</u>

résolution d'un système linéaire

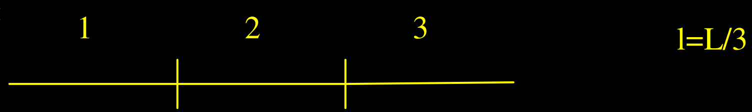
*Cholesky
*Methode du gradient conjugué

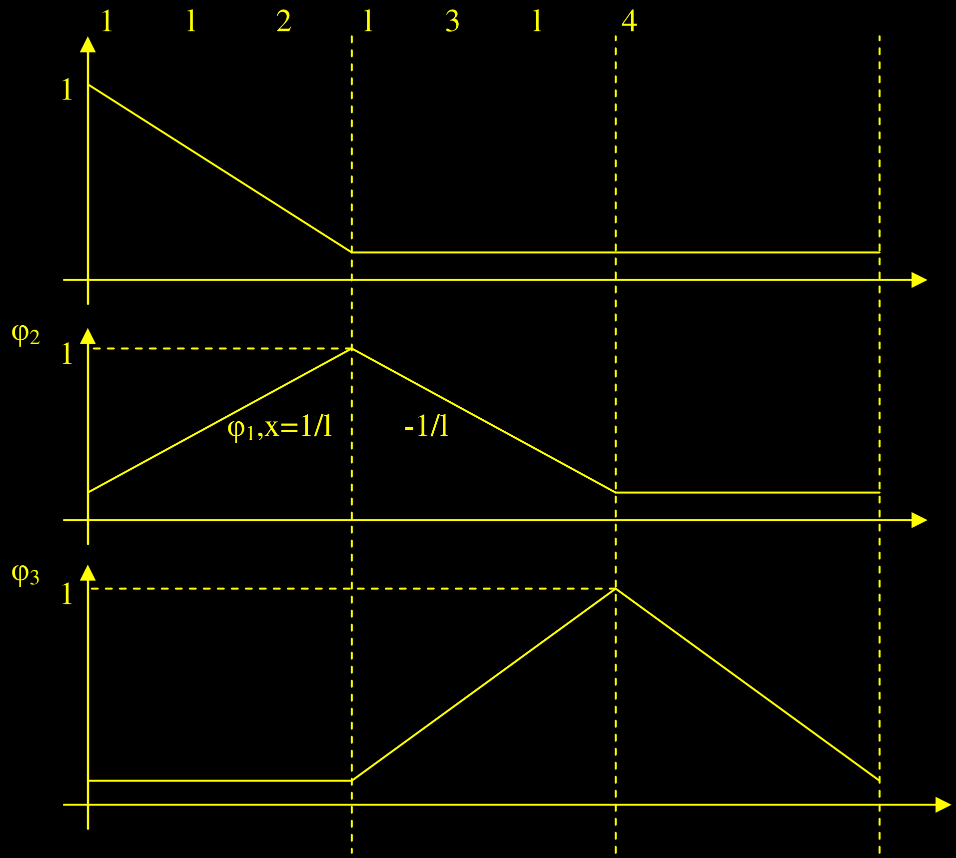
c) Méthode des éléments finis en 1D

Ex:



Calcul EF:





$U_n(x) = U_1\,\varphi_1(x) + U_2\,\varphi_2(x) + U_3\,\varphi_3(x) + U_4\,\varphi_4(x)$

U_1, U_2, U_3, U_4

E_p avec U_n CA => $U_1 = 0\ \forall\ (u_1,u_2,u_3)$
$E_p(U_n) = 1/2\ {}_0\!\int^L ES\ U_{n,x}\,dx - {}_0\!\int^L f_d\ U_n(x)\,dx$

Un(x)

$N = ESU_{,x}$

$E_d = 1/2\ {}_0\!\int^L N\varepsilon_x\,dx$
$\quad = 1/2\ {}_0\!\int^L N^2/ES\,dx$

$E_d = 1/2 \int Tr\,[\varepsilon\ K\ \varepsilon\,]\,d\Omega$

$$E_d(U_n) = 1/2\ {}_0\!\int^L ES\ \Big(\sum_{i=2}^{n} u_i\,\varphi_{i,x}\Big)\Big(\sum_{j=2}^{n} u_j\,\varphi_{j,x}\Big)\,dx$$

$U_n = U_1\,\varphi_1(x) + U_2\,\varphi_2(x) \qquad U_{x,x} = U_1\,\varphi_{1,x} + U_2\,\varphi_{2,x}$
$\quad = 1/2 \sum u_i\,u_j\ {}_0\!\int^L ES\ \varphi_{i,x}\,\varphi_{j,x}\,dx$

$K_{22} = ES\ {}_0\!\int^L \varphi_{2,x}{}^2\,dx$
$\quad = E\ {}_0\!\int^l (1/l)^2\,dx + {}_l\!\int^{2l} (-1/l)^2\,dx + 0$

$K_{21} = 2ES/l$

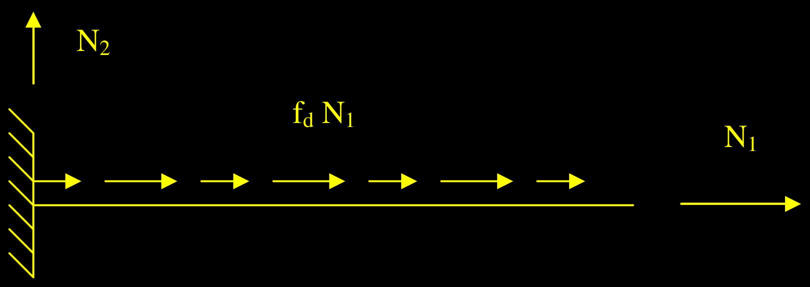
$K_{23} = Es\ {}_l\!\int^{2l} (-1/l)\ 1/l\ dl$
$\quad = F - S / l$

$K_{24} = 0$

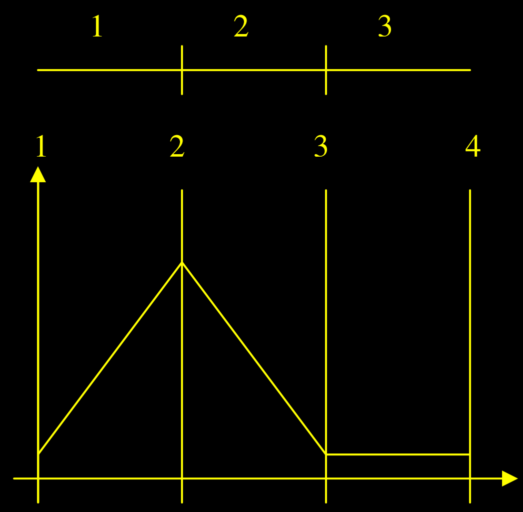
$$K = \begin{vmatrix} & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & \end{vmatrix}$$

II) Introduction à la méthode des elements finis

1) Mef 1d Poutre en traction



Trouver U(x) et N(x)
$N = \sigma_{xx} S$



$$U(x) = U_1\,\varphi_1(x) + U_2\,\varphi_2(x) + U_3\,\varphi_3(x) + U_4\,\varphi_4(x)$$

$$E_p(U_i) = 1/2 \int_0^L ES\, U_{i,x}{}^2\, dx - \int_0^L f_d\, U\, dx$$
$$= 1/2\, [U_2, U_3, U_4]\, K \begin{vmatrix} U_2 \\ U_3 \\ U_4 \end{vmatrix} - [U_2, U_3, U_4]\, F$$

où

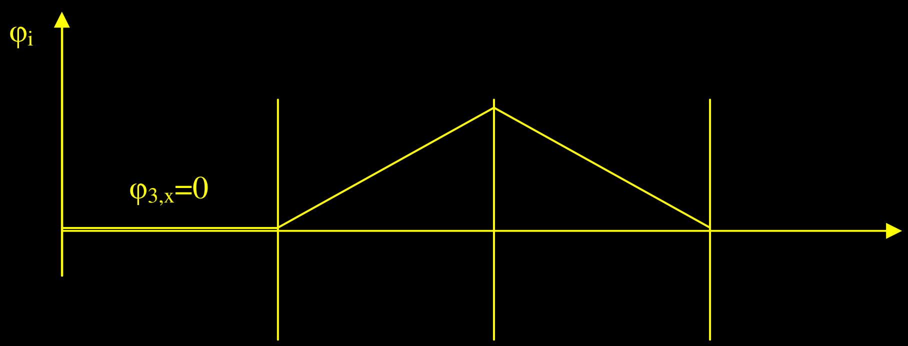
$$K_{ij} = \int_0^L ES\, \varphi_{i,x}\, \varphi_{j,x}\, dx$$

et

$$Fi = \int_0^L f_d\, \varphi_e\, dx$$

ex :

K_{23}



$K_{23} = \int_0^L ES\, \varphi_{2,x}\, \varphi_{3,x}\, dx = ES \int_l^{2l} (-1/l)(1/l)\, dx$

$K_{23} = -ES/l$

$K_{24} = 0$

$$K \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} \quad \text{et} \quad F \begin{vmatrix} \\ \\ \\ \end{vmatrix}$$

$F_1 = f \; {}_0\!\int^L \varphi_2 \, dx = f_d \, l \left| \begin{matrix} 1 \\ 1 \\ 1/2 \end{matrix} \right.$

$E_p(U_2, U_3, U_4) = 1/2 \; U^T K U - U^T F$

Min E_p / U $\quad K U - F = 0 \quad \Rightarrow K U = F$

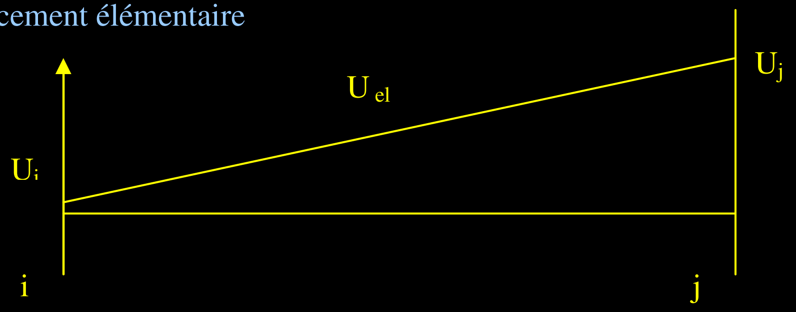
En réalité (/ code EF)

E_d Totale $= \Sigma \, E_d$ el

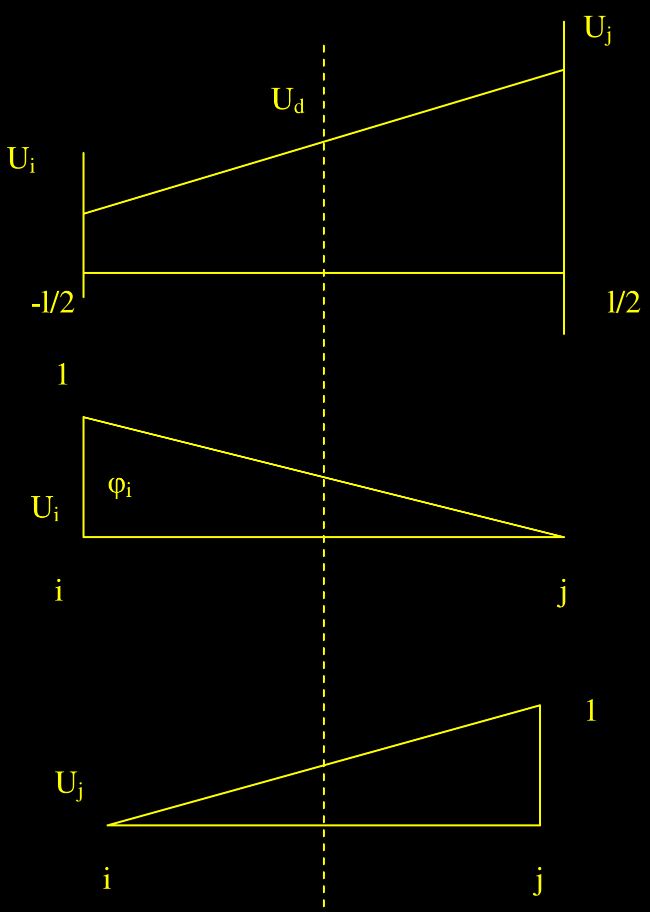
$= 1/2 \; {}_0\!\int^L ES \, (U_{,x})^2 \, dx$

$= \Sigma \; 1/2 \; {}_{el}\!\int ES \, (U_{el,2})^2 \, dx$

Déplacement élémentaire



$\forall$ i,j $\quad U_{el} =$



$\varphi_i = -1/l \; x + 1/2$

$\varphi_j = 1/l \; x + 1/x$

$U_{el} = U_i \, \varphi_i + U_j \, \varphi_j$

$= U_i \, (-1/lx + 1/2) + U_j (1/2x + 1/2)$

E_d el = 1/2 $_{-l/2}\int^{l/2}$ ES ($U_{,x}$ el)² dx
= 1/2 ES $_{-l/2}\int^{l/2}$ ($-U_i/l + U_j/l$)² dx
= 1/2 ES/l ($U_i^2+U_j^2-2U_iU_j$)

$$= 1/2\,[U_i, U_j]\, ES/l \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} U_i \\ U_j \end{vmatrix}$$

E_d Tot = $Ed_1 + Ed_2 + Ed_3$

$$E_d \text{ tot} = 1/2\, ES/l\, \left[U_2.1.U_2 + [U_2,U_3] \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} . \begin{vmatrix} U_1 \\ U_3 \end{vmatrix} + [U_3,U_4] \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} U_3 \\ U_4 \end{vmatrix} \right]$$

$$= 1/2\,[U_2\ U_3\ U_4]\, ES/l\ . \begin{vmatrix} 1+1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{vmatrix} . \begin{vmatrix} U_2 \\ U_3 \\ U_4 \end{vmatrix}$$

$$= 1/2\,[U_2\ U_3\ U_4]\, ES/l \begin{matrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{matrix} \begin{matrix} U_2 \\ U_3 \\ U_4 \end{matrix}$$

$$= 1/2\,[U_2\ U_3\ U_4]\, K \begin{vmatrix} U_2 \\ U_3 \\ U_4 \end{vmatrix}$$

$$\text{avec } K = ES/l \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

De même pour le travail des efforts extérieurs

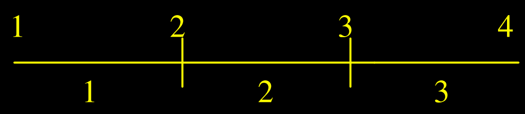
$_0\int^L f_d\, U\, dx = \Sigma_{el}\ {}_{el}\int f_d\, U_{el}\, dx$

$= \Sigma_{el}\, (U_{el})^T F_d$

$_{el}\int f_d\, U_{el}\, dx = f_d\ {}_{-l/2}\int^{l/2} [\, U_i\, \varphi_i(x) + U_j\, \varphi_j(x)\,]\, dx$

$= f_d\, [\, U_i\, l/2 + U_j\, l/2]$

$$= [\, U_i\ U_j]\, f_d/2\ l \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$



$$U^T F = [\, U_2\ U_3\ U_4]\ f_d\ l/2 \begin{vmatrix} 1+1 \\ 1\ -1 \\ 1 \end{vmatrix}$$

Min E_p => K U = F

$$ES/l \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} U_2 \\ U_3 \\ U_2 \end{vmatrix} = f_d l \begin{vmatrix} 1 \\ 1 \\ 1/2 \end{vmatrix}$$

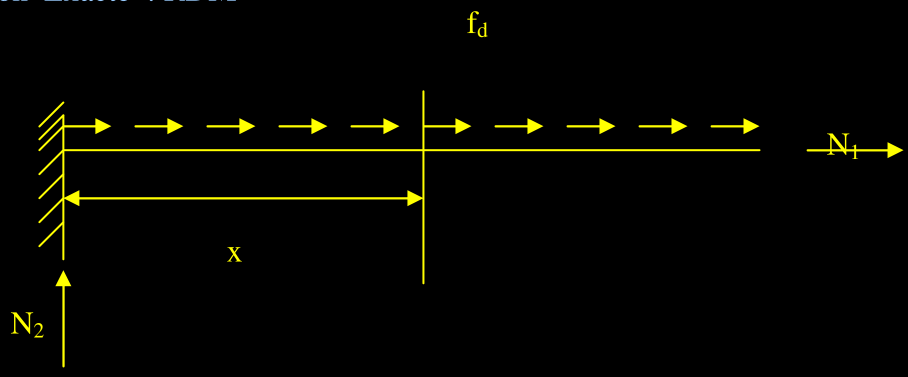
Σ ligne => $U_2 = 5/2\ f_d\ l^2 / ES$

$U_3 = 4\ f_d\ l^2/ES$

$U_4 = 9/2\ f_d\ l^2 / ES$

Solution
Element
Finis
3 (éléments)

Solution Exacte : RDM



$dR / dx + R_d = 0$

$dN / dx_1 + f_d = 0$

$N = - f_d\ x + cste$

$N(L) = 0 = /f_d L + cste$

$N = f_d\ (L - x)$

$= ES\ U_{,x}$

$= f_d\ (L - x)$

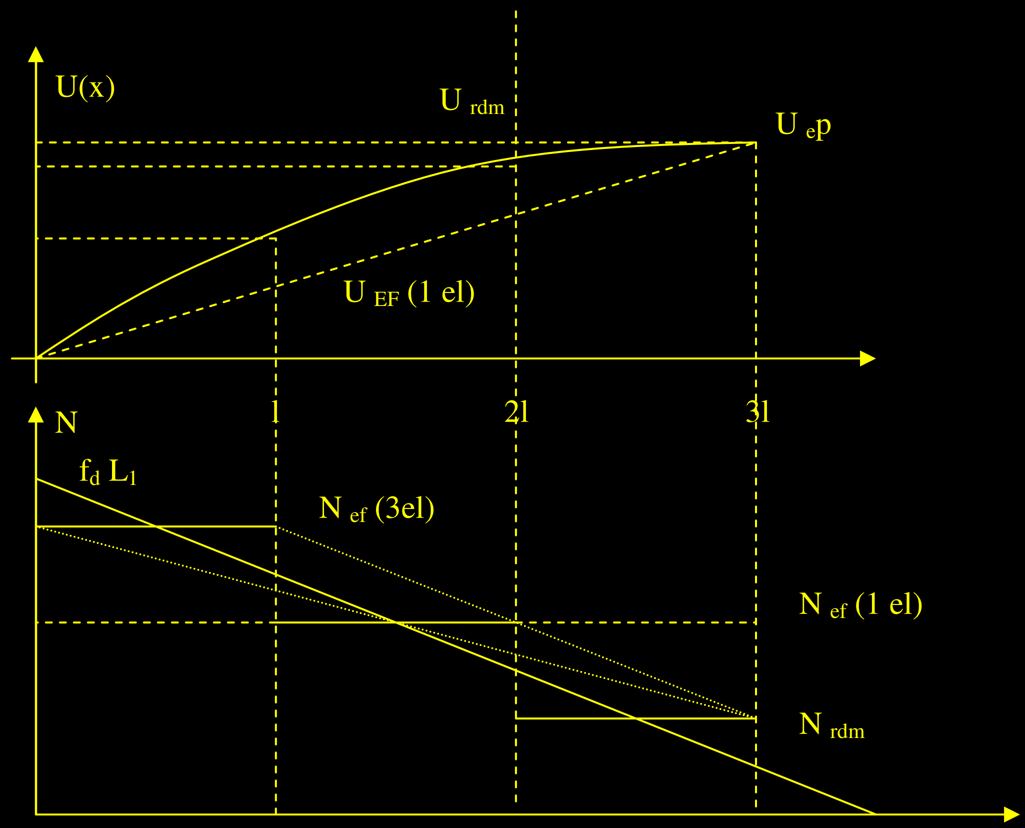
$ES\ u = f_d\ (Lx - x^2/2) + cste$

$U(0) = 0 = cste$

$U(x) = f_d/ES\ (\ 3lx - x^2/2)$

Remarque

$U_{rdm}(l) = 5/2\ f_d\ l^2 / ES = U_2^{EF}$
$U_{rdm}(2l) = 4\ f_d l^2/ES = U_3^{EF}$
$U_{rdm}(3l) = 9/2\ f_d\ l^2 / ES = U_4^{EF}$

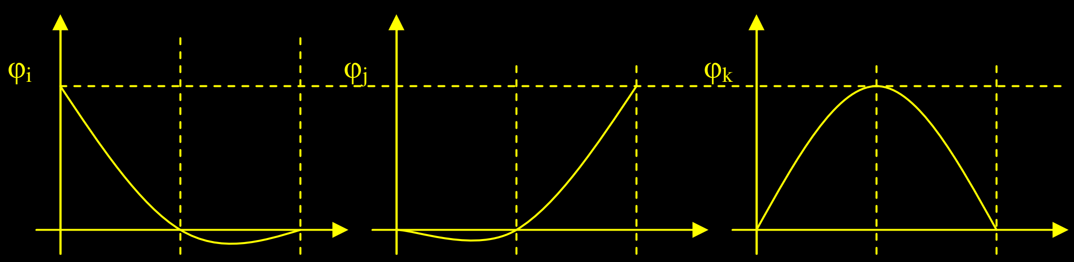


δ caractérise la qualité de la solution E F
Si δ est grand => raffiner le maillage (mettre plus d'éléments)

2 Eléments Quadratique

$U_{el} = ax^2 + bx + c$
$U_{el} = ax + b$

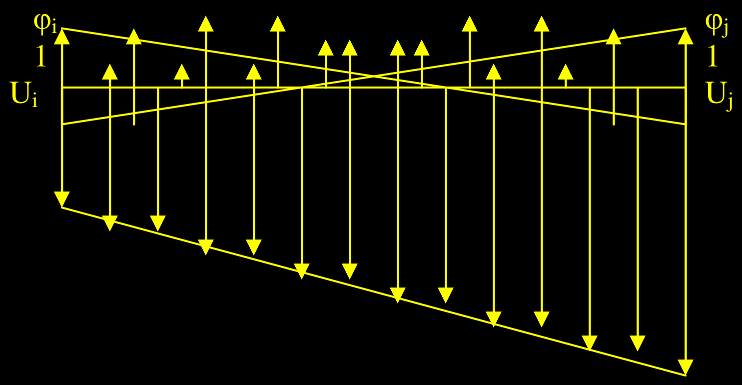
$U_{el} = ax^2 + bx + c$
$U_{el} = U_i\ \varphi_i(x) + U_j\ \varphi_j(x) + U_k\ \varphi_k(x)$



$\varphi_i\,(-l/2) = 1 = a_i\,(l/2) + k_i\,(-l/2) + C_i$
$\varphi_i\,(0) = 0 = a_i + + b_i + 0 + C_i$
$\varphi i\,(l/2) = 0 = ai\,(l/2)^2 + bi\,(l/2) + Ci$

<u>Eléments lineaires</u>

Remarque



$U(x) = U_i\,\varphi_i\,(x) + U_j\,\varphi_j\,(x)$

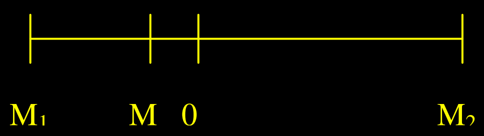
Les fonctions φ_i sont les coordonnées barycentriques en 1D

$\lambda_1\,(M_1) = 1$
$\lambda_2\,(M_1) = 0$

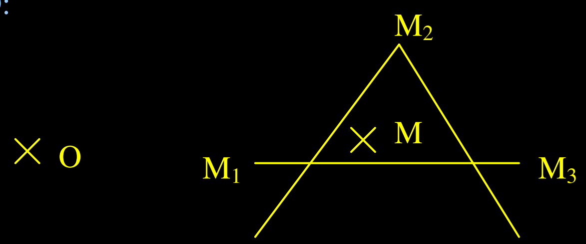
$\lambda_1(M_2) = 0$
$\lambda_2(M_2) = 1$

$OM = OM_1\,\varphi_1(x) + OM_2\,\varphi_2(x)$

φ_1 , φ_2 ou λ_1 et λ_2



En 2D:



$OM = OM_1\,\lambda_1\,(x,y) + OM_2\,\lambda_2\,(x,y) + OM_3\,\lambda_3\,(x,y)$

ex:

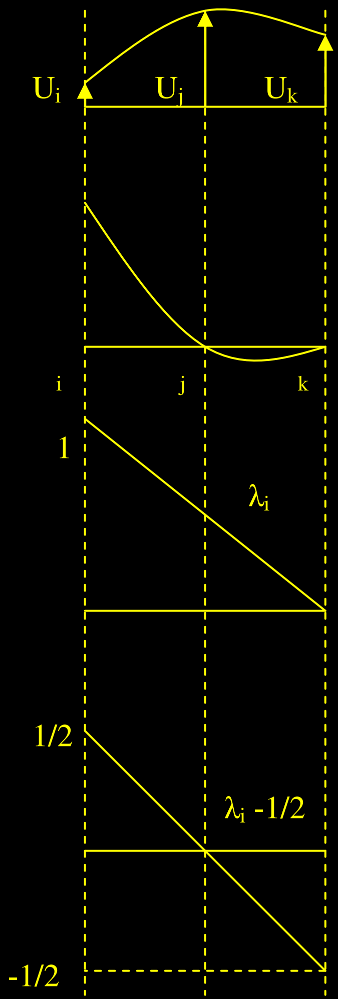
$M \Rightarrow M_1$	$\lambda_1 = 1$	$M \Rightarrow M_2$	$\lambda_1 = 0$	$M \Rightarrow M_3$	$\lambda_1 = 0$
	$\lambda_2 = 0$		$\lambda_2 = 1$		$\lambda_2 = 0$
	$\lambda_3 = 0$		$\lambda_3 = 0$		$\lambda_3 = 1$

⇨ $\lambda_1 = a_1 x + b_1 y + C_1$

III) Elements quadratiques: Polynômes de degrés 2

$U(x) = ax_2 + bx + c$

3 Coefficients => 3 nœuds

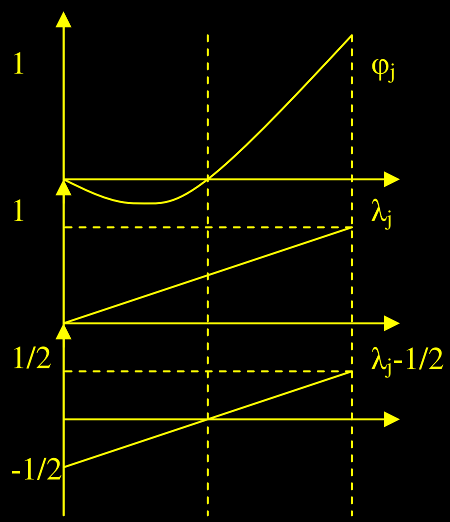


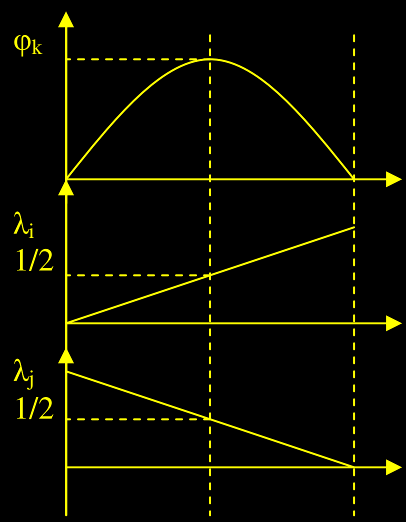
$U(x) = U_i\ \varphi_i + U_j\ \varphi_j + U_k\ \varphi_k$

φ_i, φ_j, φ_k, fonction de °2

$\varphi_i\ (-l/2) = 1$ $\qquad\qquad$ $\varphi_i = a_i\ x^2 + b_i\ x + c$
$\varphi_j\ (0) = 0$ $\qquad\qquad$ $d°2 = d°1 \times d°1$
$\varphi_k = (l/2) = 0$

$\varphi_i = 2\ \lambda_i\ (\ \lambda_i\ -1/2\ )$
$\varphi_j = 2\ \lambda_j\ (\ \lambda_j - 1/2)$
$\varphi_k = 4\ \lambda_i\ \lambda_j$





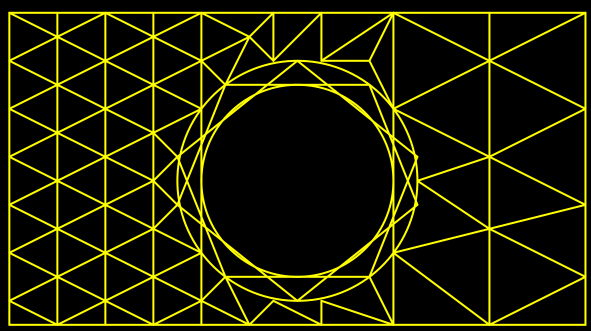
$$Ed_{el} = 1/2 \int_{-l/2}^{l/2} ES\ (U_{,x})^2\ dx$$
$$= 1/2 \int_{-l/2}^{l/2} ES\ (U_i\ \varphi_{i,x} + U_j\ \varphi_{j,x} + U_k\ \varphi_{k,x})^2\ dx$$
$$= 1/2 [\ U_i\ U_j\ U_k]\ K^{el} \begin{vmatrix} U_i \\ U_j \\ U_k \end{vmatrix}$$

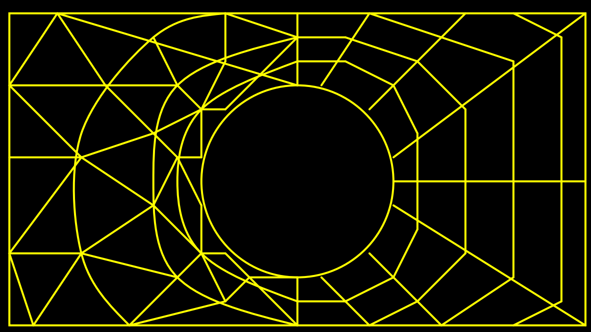
$$K^{el} = ES/\ 3.l \begin{vmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ 8 & -8 & 16 \end{vmatrix}$$

IV) Méthode des éléments finis en 2D en élasticité

(Pb en contraintes planes, Déformations planes)

$$U(x,y,z) \Rightarrow U(x,y) \left| \begin{array}{l} u(x,y) \\ v(x,y) \end{array} \right.$$





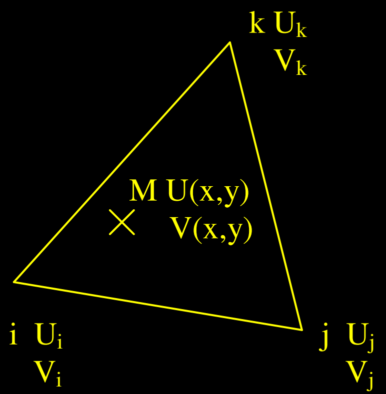
$U(x,y) = \Sigma\, U_i\, y_{,i}(x,y) \Rightarrow EP^T$

On écrit l'Ep par élément

$EP^T = \Sigma\, Ep^{el} \Rightarrow Ed^{el}$

1) Elements triangulaires bilinéaires

⇨ polynôme de degrès 1 => P1



U^{el} (x,y) = U(x,y) = $u_i\, \varphi_i(x,y) + u_j\, \varphi_j(x,y) + u_k\, \varphi_k(x,y)$
V(x,y) = $v_i\, \varphi_i(x,y) + v_j\, \varphi_j(x,y) + v_k\, \varphi_k(x,y)$

Fonction de base

$\varphi_i(x,y) = \lambda_i(x,y)$
$\varphi_i(x_i,y_i) = 1$
$\varphi_i(x_j,y_j) = 0$
$\varphi_i(x_k,y_k) = 0$

$\varphi_i(x,y) = a_i\, x + b_i\, y + c_i$
$\varphi_i(x,y) = 1/\Delta\ [\ (y_i - y_k)\, x - (x_j - x_k)y + x_j\, y_k - x_k\, y_j$
avec $\Delta = (x_i - x_j)(y_j - y_k) - (x_j - x_k)(y_i - y_j)$

Propriété

Δ = 2 aires du triangle

Matrice de rigidité élémentaire

$E_p (U)$

$E_d^{el} (U) = 1/2 \int Tr [\sigma (u) \varepsilon (u)] d\Omega$

1) Déformation plane

$\varepsilon_{zz} = 0$

$\sigma = \lambda\, tr [\varepsilon] Id + 2\mu\, \varepsilon$

en 3D en 2D

$tr [\varepsilon 3D] = tr [\varepsilon 2D] = \varepsilon xx + \varepsilon yy$

$$\varepsilon^{2D} = \begin{matrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{matrix}$$

$$= \begin{matrix} U_{,x} & 1/2(U_{,x} + V_{,y}) \\ 1/2(U_{,x} + V_{,y}) & V_{,y} \end{matrix}$$

$$= \begin{matrix} u_i\, \varphi_i(x,y) + u_j\, \varphi_j (x,y) + u_k\, \varphi_k (x,y) & 1/2[u_i\, \varphi_i(x,y) + u_j\, \varphi_j (x,y) + u_k\, \varphi_k (x,y) \\ & - v_i\, \varphi_i(x,y) + v_j\, \varphi_j (x,y) + v_k\, \varphi_k (x,y)] \\ 1/2[u_i\, \varphi_i(x,y) + u_j\, \varphi_j (x,y) + u_k\, \varphi_k (x,y) & v_i\, \varphi_i(x,y) + v_j\, \varphi_j (x,y) + v_k\, \varphi_k (x,y) \\ - v_i\, \varphi_i(x,y) + v_j\, \varphi_j (x,y) + v_k\, \varphi_k (x,y)] & \end{matrix}$$

D.p .

$E_d{}^{el} = 1/2 \int (\sigma_{xx} (\varepsilon) \varepsilon_{xx} + \sigma_{yy} (\varepsilon) \varepsilon_{yy} + 2\, \sigma_{xy}\, \varepsilon_{xy}) ds$

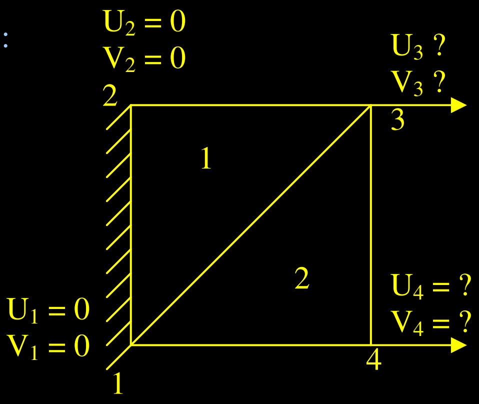
$\sigma_{xx} = ((\lambda + 2\mu) \varepsilon_{xx} + \lambda\, \varepsilon_{yy})$
$\sigma_{yy} = ((\lambda\, \varepsilon_{xx} + (\lambda + 2\mu) \varepsilon_{yy})$
$\sigma_{xy} = (2\, \mu\, \varepsilon_{xy})$

$$ED^{el} = 1/2\, {}_T\!\int [(\lambda + 2\, \mu) (\varepsilon_{xx}{}^2 + \varepsilon_{yy}{}^2) + 2\lambda\, \varepsilon_{xx}\, \varepsilon_{yy} + 4\mu\, \varepsilon_{xy}{}^2] ds$$

$$= 1/2 [U_i, V_i, U_j, V_j, U_k, V_k] \quad \underset{\substack{x_i,x_j,x_k \\ y_i,y_j,y_k}}{K^{el}} \quad \begin{matrix} U_i \\ V_i \\ U_j \\ V_j \\ U_k \\ V_k \end{matrix}$$

$$K_{11} = {}_T\int [(\lambda + 2\mu)\, \varphi_{i,x}^{\,2} + 4\,\mu\ 1/4\, (\varphi_{i,j})^2]\, ds$$

ex :



$$Ed_1 = 1/2\, [\, U_3\ V_3\,]\, [\, K_1\,] \begin{matrix} U_3 \\ V_3 \end{matrix}$$

$$Ed_2 = 1/2\, [\, U_3, V_3, U_4, V_4\,]\, [\, K_2\,] \begin{matrix} U_3 \\ V_3 \\ U_4 \\ V_4 \end{matrix}$$

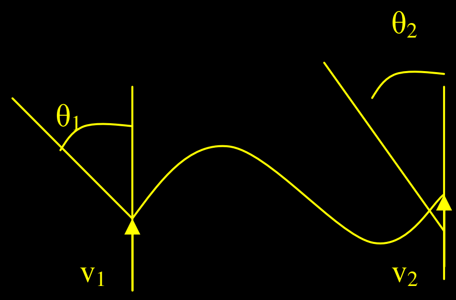
$$Ed^T = Ed_1 + Ed_2$$

$$= 1/2\, [\, U_3, V_3, U_4, V_4\,] \begin{matrix} K_1 & \\ & \\ & \\ & K_2 \end{matrix} \quad \begin{matrix} U_3 \\ V_3 \\ U_4 \\ V_4 \end{matrix}$$

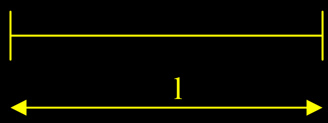
$$U^T F = [\, U_3, V_3, U_4, V_4\,] \begin{matrix} F \\ 0 \\ F \\ 0 \end{matrix}$$

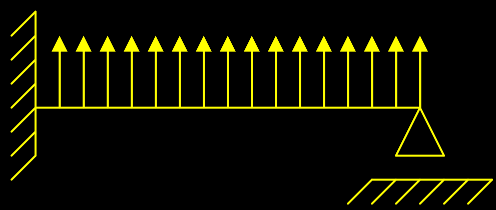
Minimum Ep ⇔ KU=F

$$[K°] = 2EI / l^3 \begin{bmatrix} 6 & & & \\ 3l & 2l^2 & & \\ 6 & -3l & 6 & \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

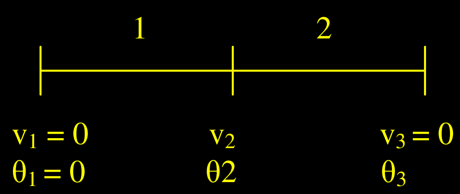


$$F^{el} = P/2 \begin{bmatrix} l \\ l^2/6 \\ l \\ l^2/6 \end{bmatrix}$$





$$W_{el} = (\quad)(F_{el})$$



$$E_d = 1/2\,(v_1, \theta_1, v_2, \theta_2)\,[K] \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

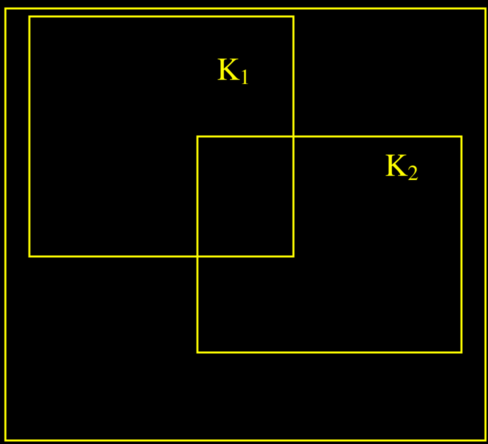
$$2EI/l^3 \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = P/2 \begin{bmatrix} \quad \end{bmatrix}$$

$$Ed = Ed_1 + Ed_2 = 1/2\,(0 \;\; 0 \;\; v_2 \; \theta_2)\,[K_1] \begin{bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \end{bmatrix} + 1/2\,(v_2 \; \theta_2 \; 0 \; \theta_3)\,[K_2] \begin{bmatrix} v_2 \\ \theta_2 \\ 0 \\ \theta_3 \end{bmatrix}$$

$$W_{ext} = W_{ext\,1} + W_{ext\,2}$$
$$= [K_{33} v_2^2 + 2 K_{34} v_2 \theta_2 + K_{44} \theta_2^2]$$

$$Ed = 1/2\,(\,0\;0\;v_2\;\theta_2\;0\;\theta_3)\,[\,K\,G\,]\begin{bmatrix}0\\0\\v_2\\\theta_2\\0\\\theta_3\end{bmatrix}$$

$2EI/l^3$ *



$$\begin{vmatrix}v_1\\\theta_1\\v_2\\\theta_2\\v_3\\\theta_3\end{vmatrix} = P/2\begin{pmatrix}F_1\\ \\F_2\end{pmatrix}$$

$$2EI/l^3\begin{bmatrix}6+6 & -3l+3l & 3l\\ -3l+3l & 2l^2+2l^2 & l^2\\ 3l & l^2 & 2l^2\end{bmatrix} = P/2\begin{pmatrix}l + l\\ \frac{-l^2}{6} + \frac{l^2}{6}\\ -\frac{l^2}{6}\end{pmatrix}$$

$$\frac{2EI}{l^3} * \begin{matrix}12 & 0 & 3l\\ 0 & 4l^2 & l^2\\ 3l & l^2 & 2l^2\end{matrix}\begin{pmatrix}v_2\\\theta_2\\\theta_3\end{pmatrix} = P/2\begin{pmatrix}2l\\0\\-l^2/6\end{pmatrix}$$

$4\,\theta_2 + \theta_3 = 0 \quad \Rightarrow \quad \theta_2 = -\,\theta_3\ /\ 4$

$12\,v_2 + 3l\,\theta_3 = Pl^4\ /\ 2EI$
$3l\,v_2 + l^2\,\theta_2 + 2l^2 = -\,Pl^5\ /\ 24\,EI$

$12\,v_2 + 3l\,\theta_3 = Pl^4\ /\ 2\,EI$
$3\,v_2 + l\theta_2 + 2l\,\theta_3 = -\,Pl^4\ /\ 24\,EI$

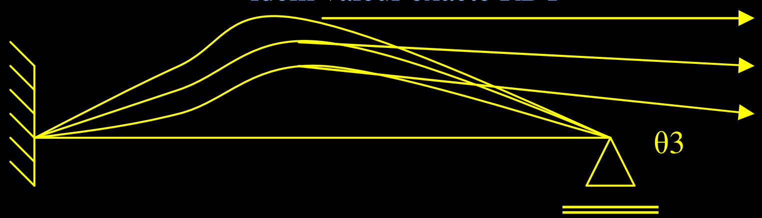
$12\,v_2 + 3l\,\theta_3 = Pl^4\ /\ 2\,EI$
$3\,v_2 + 7l\,\theta_3\ /\ 4 = -\,Pl^4\ /\ 24\,EI$

$12\,v_2 + 3l\,\theta_3 = Pl^4\ /\ 2\,EI$
$12\,v_2 + 7l\,\theta_3 = -\,Pl^4\ /\ 6\,EI$

$4\,l\,\theta_3 = -2/3\,Pl^4\ /\ EI$

$\theta_3 = -1/6\,Pl^3\ /\ EI$
$= -\,Pl^3\ /\ 48\,EI$

% maillage à 1 élément : idem
idem valeur exacte RD1



Exacte °4
2 éléments ; 2 fonctions du 3° degrés
1 élément : 1 fonction du 3° degrès
$\theta 3$

Remarque :

[K°]

F

$W_{ext} = F\,\theta_2 + M\theta_1$

Section EF avec 1 élément

$F_e = \begin{matrix} 0 \\ 0 \\ F_0 \end{matrix}$

$v_1 = 0$ $\qquad$ $v_2 = ?$
$\theta_2 = 0$ $\qquad$ $v_3 = ?$

$$\frac{2EI}{l^3}\ \boxed{\begin{matrix} 6 & -3l \\ -3l & 2l^2 \end{matrix}}\ \begin{pmatrix} \\ \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F \\ M \end{pmatrix}$$

Si $M = 0 \Leftrightarrow \begin{cases} 6\,v2 - 3l\ \theta_2 = Fl^3 / 2EI \\ -3lv2 + 2l^2\ \theta_2 = 0 \end{cases}$

$\Rightarrow\ l\,\theta_2 = 3/2\ v_2$

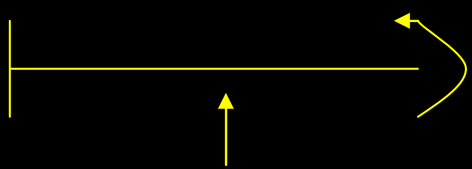
$\Rightarrow\ v_2 = Fl^3 / 3\,EI$

$dT_2/dx + 0 = 0$

$dMf_3 / dx + T_2 = 0$

$Mf_3 = EI\ v''_2$

Sections EF exactes confondues

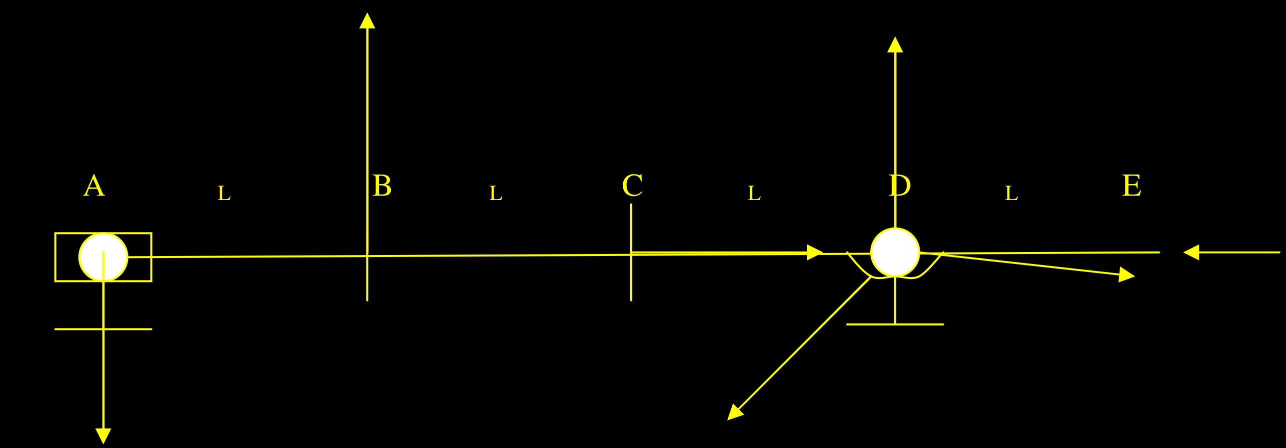


Maillage 2 éléments => solution exacte

<u>Exam de mécanique des structures</u>

<u>Partie 1: Etude de l'arbre en résistance des matériaux</u>

a) déterminer les efforts de liaison en fonction de Cb ,M, L et α



$$\left\{\begin{array}{cc} X_A & 0 \\ Y_A & 0 \\ 0 & 0 \end{array}\right\}_A \quad \left\{\begin{array}{cc} F_T & 0 \\ F_N & 0 \\ 0 & C_B \end{array}\right\}_B \quad \left\{\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ M_g & 0 \end{array}\right\}_C \quad \left\{\begin{array}{cc} X_D & 0 \\ Y_D & 0 \\ Z_D & 0 \end{array}\right\}_D \quad \left\{\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & N_E \end{array}\right\}_E$$

$\alpha = 20°$

$F_T = CB / R$

$F_N = F_T \tan \alpha$

PFS :

en D

$X_A + F_T + X_D = 0$

$Y_A + F_N + Y_D = 0$

$M_g + Z_d = 0$

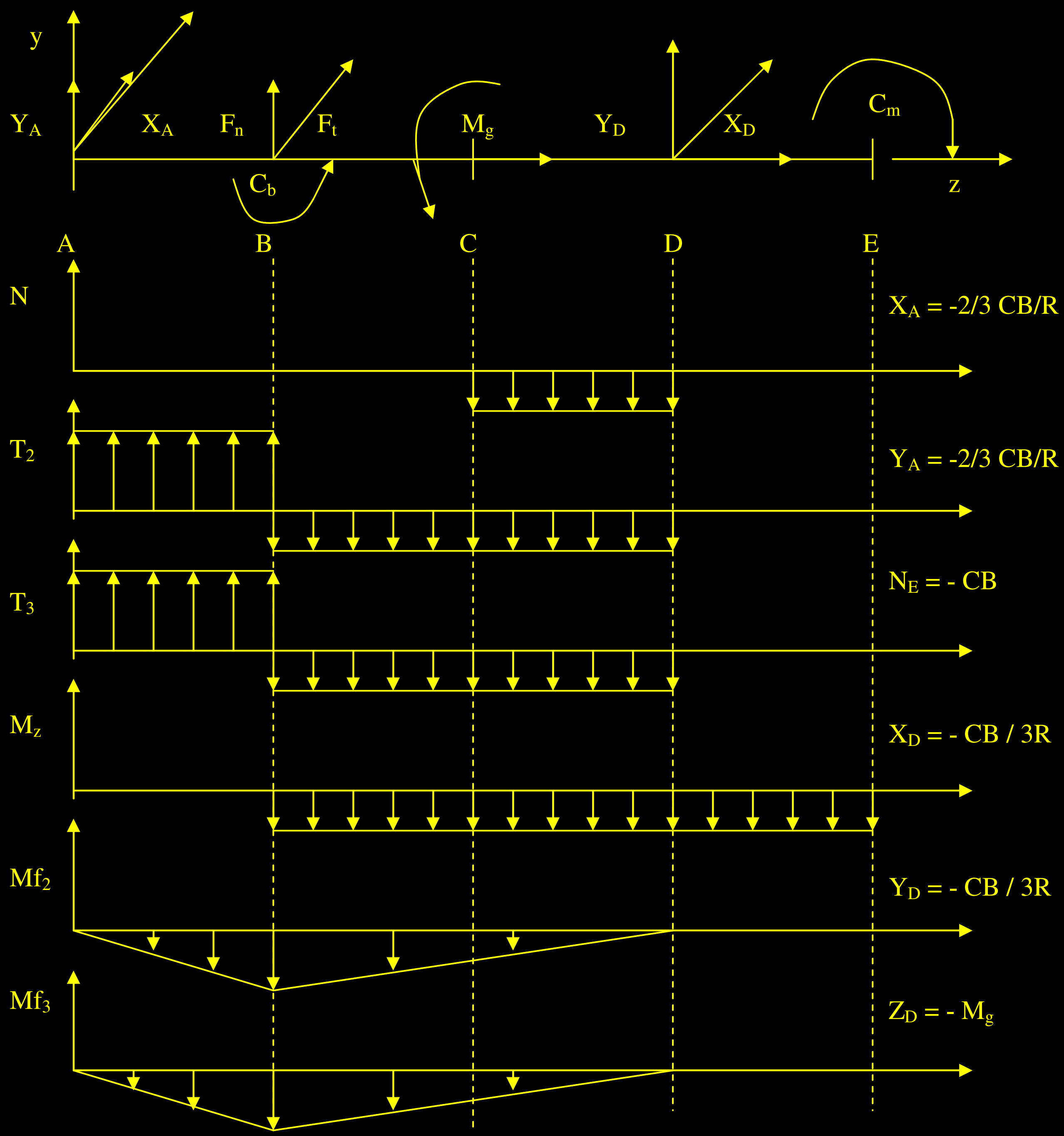
$DA \wedge R_A + DB \wedge F_B + DC \wedge R_C + CB\ z + N_E\ z = 0$

$$\left|\begin{matrix}0\\0\\-3L\end{matrix}\right. \wedge \left|\begin{matrix}XA\\YA\\0\end{matrix}\right. + \left|\begin{matrix}0\\0\\-2l\end{matrix}\right. \wedge \left|\begin{matrix}FT\\FN\\0\end{matrix}\right. + \left|\begin{matrix}0\\0\\-L\end{matrix}\right. \wedge \left|\begin{matrix}0\\0\\Mg\end{matrix}\right. + \left|\begin{matrix}0\\0\\CB\end{matrix}\right. + \left|\begin{matrix}0\\0\\N_E\end{matrix}\right. = 0$$

$3L\ Y_A + 2L\ F_N = 0$
$-3L\ X_A - 2L\ F_T = 0$
$CB + N_E = 0$

$Y_A = -2/3\ CB/R \tan\alpha$		$X_D = -1/3\ CB/R$
$X_A = -2/3\ CB/R$	et	$Y_D = -1/3\ CB/R \tan\alpha$
$Ne = -CB$		$Z_D = -M_g$

b) Efforts intérieurs



b) $Mf_3{}^2$ maxi

c) $Mf_3{}^2$ 2x + faible

$(MF_2{}^2 + Mf_3{}^2)\ 1/2 \Rightarrow M_F$

$U(r)\ e_r = a\Omega e_r$

MF r => N

- Contraintes planes rete sur r
- axisymétrique (r, ω, z)

$U(r)\ e_r = a\ \Omega\ e_r$

$E_p = E_p - W_{ext}$
$= 1/2\ \rho\ 2\Pi\ 0\int R\ \ tr\ (\varepsilon\ K\ \varepsilon)\ r\ dr\ - e\ 2\Pi\ {}_0\int R\ (\rho\omega^2 r)\ (a_r)\ rdr$

Notation ingénieur

ε , σ => ε, σ

Tenseurs ordre 2 (matrices) ordre 1 (vecteurs)

$2D\ \varepsilon \begin{matrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{matrix} \Rightarrow \varepsilon \begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{xy} \end{matrix}$ ou $\varepsilon \begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{matrix}$

$\sigma \begin{matrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{matrix} \Rightarrow \sigma \begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xy} \end{matrix}$ ou $\sigma \begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ 2\sigma_{xy} \end{matrix}$

$e_d = 1/2\ Tr\ [\ \sigma\ \varepsilon\ ] = 1/2\ \sigma : \varepsilon = 1/2\ \sigma_y''\ \varepsilon_y''$

$e_d = 1/2\ (\ \sigma_{xx}\ \varepsilon_{xx} + \sigma_{yy}\ \varepsilon_{yy} + 2\ \sigma_{xy}\ \varepsilon_{xy})$

$1/2\ \sigma\ \varepsilon = 1/2\ \sigma_i\ \varepsilon_i = 1/2\ [\ \sigma_{xx}\ \varepsilon_{xx} + \sigma_{yy}\ \varepsilon_{yy} + 2\ \sigma_{xy}\ \varepsilon_{\ xy}\ ]$

(u . v)

Relation de comportement

$\sigma = K\ \varepsilon$
$\quad = K\ \varepsilon$

$\sigma_{ij} = K_{ijkl}\ \varepsilon_{kl}$

Iso:

$\sigma = \lambda + \nu\ [\ \varepsilon\ ]\ Id + 2\mu\ [\ \varepsilon\ ]$

$K_{1111} = \lambda + 2\mu$

$K_{1112} = 0$

$K_{1122} = \lambda$

en Not ing

$\sigma = K\ \varepsilon$

$\sigma i = K_{ij}\ \varepsilon_j$

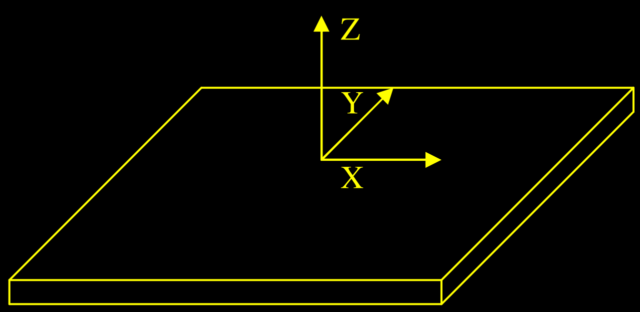
en 2D

$$\sigma \quad \begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{matrix} \quad \begin{matrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{matrix} \quad \begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{matrix}$$

K : tenseur de hooke en 2 D, notation ingénieur, déformation plane

En contraintes planes

(Structure 2D>>>>1D)(plaques ou coques)



$(\sigma_{ZZ}, \sigma_{XZ}, \sigma_{YZ}) \lll (\sigma_{XX}, \sigma_{YY}, \sigma_{XY})$

Relation de comportement

$Tr [\sigma_{3D}] = Tr [\sigma_{2D}] \quad \varepsilon_{ZZ} \neq 0$

$$Tr \begin{matrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{matrix} = Tr \begin{matrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{matrix} = \sigma_{xx} + \sigma_{yy}$$

en 2D ou en 3D

$\varepsilon = 1+\nu/E\ \sigma - \nu/E\ Tr [\sigma]\ Id$

et 3D

$\sigma = \lambda\ Tr [\varepsilon]\ Id + 2\mu\ \varepsilon$

en 2D :

$Tr [\varepsilon] = 1 + \nu / E\ tr\ \sigma - 2\ \nu / E\ tr [\sigma]$

$Tr [\varepsilon] = 1 - \nu / E\ tr [\sigma]$

en 2D

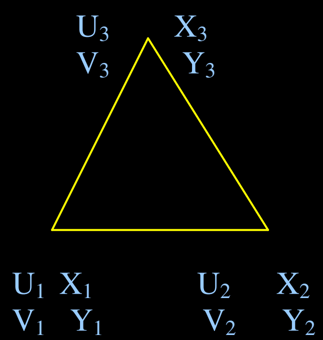
$\sigma = E/ (1 + \nu) [\varepsilon + \nu/E\ \ E/(1-\nu) \quad tr [\varepsilon]\ Id]$
$\quad = E/ (1 + \nu) [\varepsilon + \nu/(1-\nu) \quad tr [\varepsilon]\ Id]$
ou

$\sigma = E/ (1 + \nu)^2 [(1-\nu)\varepsilon + \nu/E\ \ Etr [\varepsilon]\ Id]$

ou

$$\sigma \begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{matrix} = E/(1-\nu)^2 \begin{matrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{matrix} \quad \begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{matrix}$$

M E F en 2D en contraintes planes



$$U \quad \begin{matrix} u \\ v \end{matrix} \quad \begin{matrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \end{matrix} \quad \begin{matrix} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{matrix}$$

$$U \quad \begin{matrix} u \\ v \end{matrix} \quad \begin{matrix} N^T & O^T \\ O^T & N^T \end{matrix} \quad \begin{matrix} U \\ V \end{matrix}$$

où $N \quad \begin{matrix} N_1 = \lambda_1 \\ N_2 = \lambda_2 \\ N_3 = \lambda_3 \end{matrix}$ et $U \quad \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix}$ et $V \quad \begin{matrix} V_1 \\ V_2 \\ V_3 \end{matrix}$

$$\varepsilon = \begin{matrix} \varepsilon_{xx} = U_{,x} = N^T{}_{,x} \; U \\ \varepsilon_{yy} = V_{,y} = N^T{}_{,y} \; V \end{matrix}$$

$$2\varepsilon_{xy} = (u_{,y} = v_{,x}) = N_{,y}{}^T U + N_{,x}{}^T V$$

$$\varepsilon = \begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{matrix} = \begin{matrix} N_{,x}{}^T & 0 \\ 0 & N_{,y}{}^T \\ N_{,y}{}^T & N_{,x}{}^T \end{matrix} \quad \begin{matrix} U \\ V \end{matrix} = B \quad \begin{matrix} U \\ V \end{matrix}$$

$$\varepsilon^T = [U^T , V^T] B^T$$

$ED_{el} = 1/2 \, [U^T , V^T] \, K_{el} \begin{matrix} U \\ V \end{matrix}$ objectif

$$ED_{el} = 1/2 \, {}_{el}\!\int \sigma(\varepsilon) . \varepsilon \, ds \quad (*h)$$

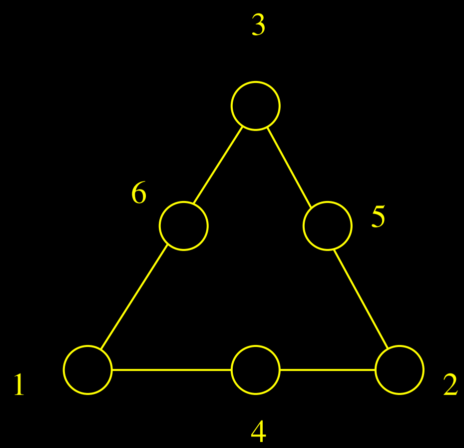
$$\begin{aligned} \sigma \, \varepsilon &= \sigma_{xx} \, \varepsilon_{xx} \\ &= \sigma^T \varepsilon \\ &= \varepsilon^T \sigma \end{aligned}$$

$$ED_{el} = 1/2 \, {}_{el}\!\int \varepsilon^T K_{cp} \, \varepsilon \, ds$$

$$1/2 \, [U^T, V^T] \, {}_{el}\!\int B^T K_{cp} B \, ds \begin{matrix} U \\ V \end{matrix}$$

$$\begin{matrix} U \\ V \end{matrix} = \begin{matrix} N^T & O^T \\ O^T & N^T \end{matrix} \quad \begin{matrix} U \\ V \end{matrix}$$

en 2D tringle à 6 nœuds $\leftarrow$ a



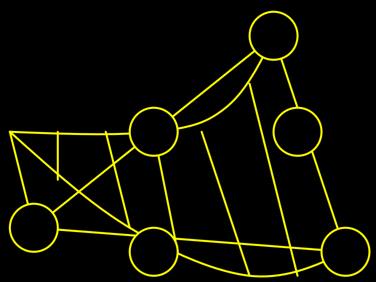
5 nœuds : base complète de degré 15

1,x,y,xy,x²,y²

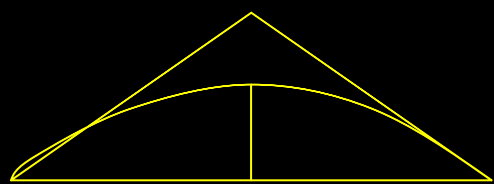
la saute $e_d = 1/2$ [U ^{T} V ^{T}] el $\int B^T K_{cp} B$ ds U
V

Ex N_1

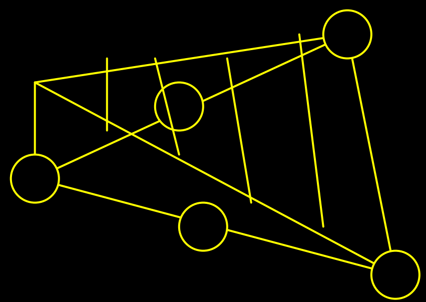
N_1



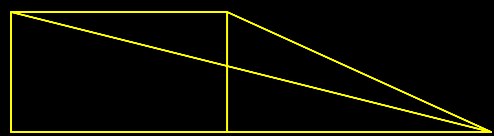
$N_1 = \lambda_1 * 2 (\lambda_1 - 1/2)$

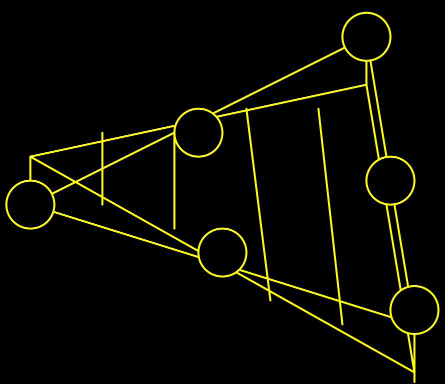


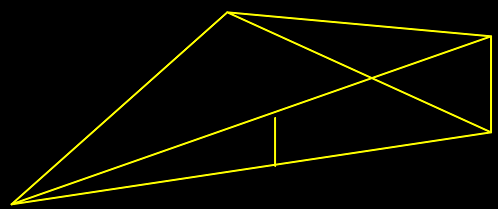
$N_4 = 4\lambda_1\lambda_2$



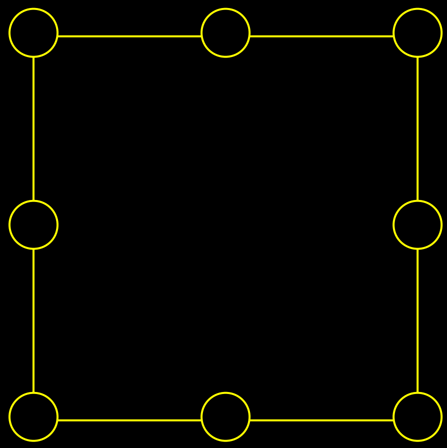
$2\ \lambda_1$





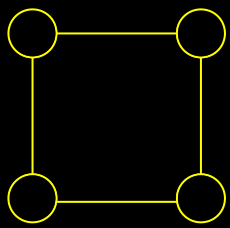


$2\ \lambda_2$

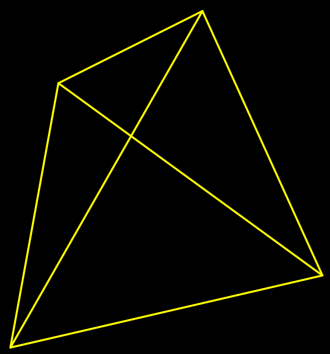


Quadrilatère à quatre nœuds ou 8 nœuds

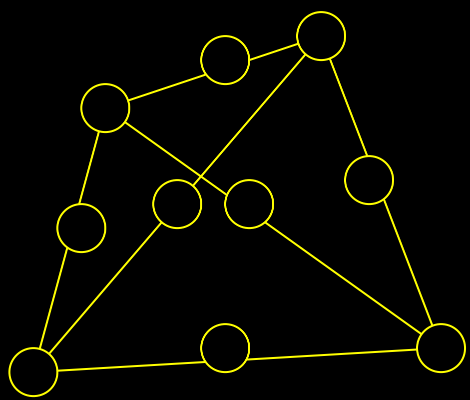
Voir cours suivant



en 3D   Tétraèdre à 4 nœuds



$4 \times 3 = 12°$



1,x,y,z   linéaire

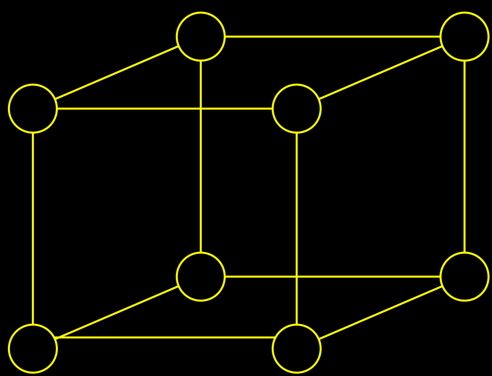
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$

tétraèdre à 10 nœuds

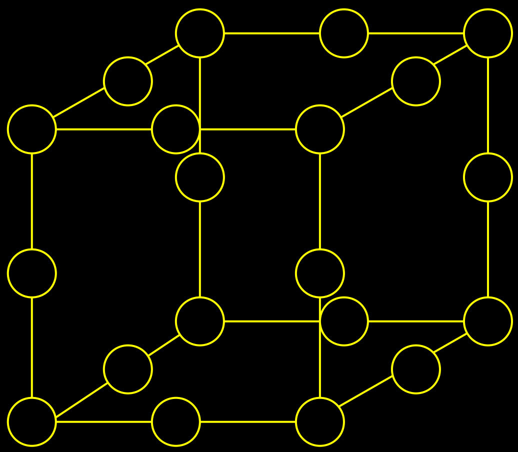
base complète de d° 2

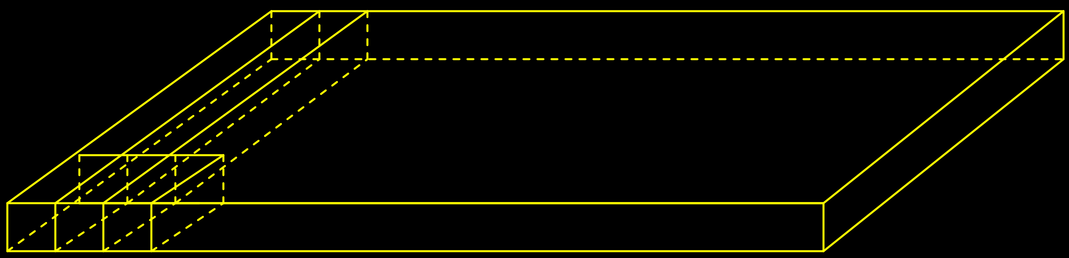
K_e : 30x30

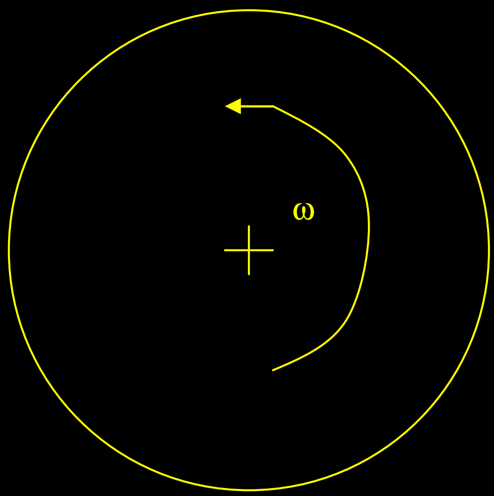
Cube à 8 nœuds



Cube à 20 nœuds







Si Ep <<< Rayon

- ⇨ Cp plan (xy)
- ⇨ hyp pb axi

$U = u(r)\, e_r$ ($\omega(r)\, e_z$) n'intervient pas

forme approchée

$U(r) = U\, R$

1°

U CA

$U(r) = 0$
$r = 0$

2°

$$E_p\ (u) = E_d\ (u^2) - W_{est}$$
$$= 1/2 \int f_r\ (\varepsilon\ (u)\ K\ \varepsilon\ (u))\ d\Omega\ _r \int f\ u\ d\ \Omega$$

$$W_{ea}L = {}_0\int^{e} {}_0\int^{2\Pi} {}_0\int^{R} (\rho\ \omega_r^2\ e_r) (u(r)\ e_r)\ \Omega\ dz\ d\theta\ dr$$

$$= \rho\ \omega^2\ a\ e\ 2\Pi\ R^4 / 4$$

$$E\ \varepsilon = (1 + \nu)\ \sigma - \nu\ (t_r\ \sigma)\ 1$$

$$\sigma_{zz} = 0 = \sigma_{rz} = \sigma_{\theta z} = 0$$

$$\sigma = \begin{array}{ccc} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{array}$$

$E_d = f(a)$ $E\,\varepsilon_{rr} = \sigma_{rr} - \gamma\,\sigma_{\theta\theta}$

$E\,\varepsilon_{\theta\theta} = \sigma_{\theta\theta} - \gamma\,\sigma_{rr}$

$E\,\varepsilon_{r\theta} = (1 + \nu)\,\sigma_{r\theta}$

et $\varepsilon = 1/2\,(\partial u/\partial M + (\partial u/\partial M)T)$

$$\partial U/\partial M = \begin{array}{ccc} \partial u/\partial r & 0 & 6 \\ 0 & U/r & 0 \\ 0 & 0 & X \end{array} \qquad u = u(r)\,e_r$$

$$\varepsilon = \begin{array}{ccc} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & x \end{array}$$

Inversion

$$\sigma_{rr} = E/\ 1-\nu^2\ (\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta})$$
$$\sigma_{\theta\theta} = E/\ 1-\nu^2\ (\varepsilon_{\theta\theta} + \nu\,\varepsilon_{rr})$$
$$\sigma_{r\theta} = E/\ 1-\nu^2\ \varepsilon_{r\theta}$$

$E_d = 1/2 \iiint E/(1-\nu^2)\,(2a^2 + 2\nu a^2)\, r\, d\theta\, dr\, dz$

$= 1/2\ E/(1-\nu)\ 2a^2\,(2\Pi)\,(e)\,R^2/2$

$W_{ext} = \rho\,\omega^2\, e\,\rho\, 2\Pi\ R^4/4$

$\partial Ep/\partial a = 0$

$= E/\ 1-\nu\quad 2a\Pi e R^2 - \rho\,\omega^2/2\ R^2\,\Pi\, e\, R^2$

$a = \rho\,\omega^2 R^2 (1-\nu)/4E$

$a = (((E/1-\nu)\,2)/(\rho\omega^2/2\ R^2))^{-1}$

$\Delta R = u(R)$

$= a\,R$

$= (\rho\,\omega^2 R^2 (1-\nu))/4E$

$\varepsilon_{zz} = \Delta l/\rho$

$E\varepsilon_{zz} = (1+\nu\ )\ \sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})$

$\varepsilon_{zz} = -\nu/E\ (\ \sigma_{rr} + \sigma_{\theta\theta})$

$\varepsilon_{zz} = -\ \nu/E\ (\sigma_{rr} + \sigma_{\theta\theta})$
$\quad = -\nu/E \quad E/(1 - \nu^2)\ (\varepsilon_{rr} + \nu\ \varepsilon_{\theta\theta} + \varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\ )$
$\quad = -\nu/(1-\nu)\ (\varepsilon\ rr + \varepsilon\ _{\theta\theta})$
$\quad = -\nu/(1-\nu)\ (2a)$
$\quad = -0{,}3/0{,}7 \quad *\ (2a)$
$\quad = -\Delta l/l$

$\Delta e = \varepsilon_{zz}\ e$
$\quad = (-\nu/1-\nu)\ \rho\ 2a$
$\quad = (-\ 2\nu\rho\ \omega^2\ R^2\ e)\ /\ 4E$

$\Delta R/\Delta e\ = ((1-\nu)R/4)/(-\nu\rho/2) = -\ 0{,}7/0{,}6 \quad R/e$

$\sigma_{ep} = \sqrt{}\sigma_{rr}{}^2 + \sigma_{\theta\theta}{}^2$

$\sigma_{rr} = E/(1-\nu)\ a = \sigma_{\theta\theta}$

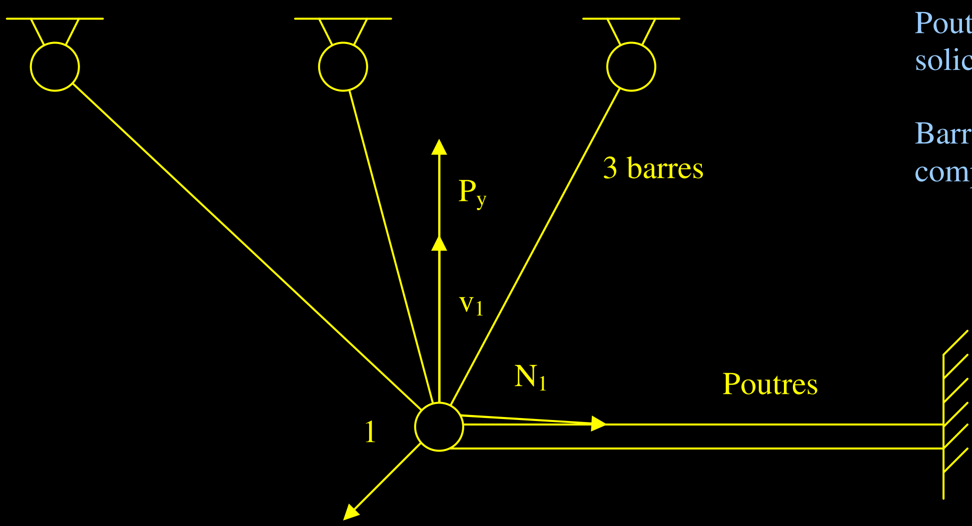
$E/(1-\nu^2)\ \ (\varepsilon_{rr} + \gamma\ \varepsilon_{\theta\theta})$

$\sigma_{eq} = \nu 2\ \ E/(1-\ \nu) \quad a$

$\rho\omega^2 R^2(\ \nu)/4E$

A.N. : $\sigma_{eq} = 61$ MPa

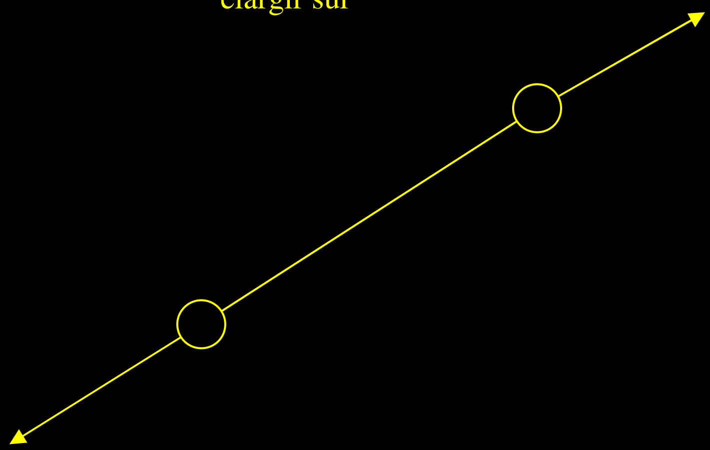
$\omega(r,z) = \varepsilon_{zz} - z \qquad \varepsilon_{zz} = \rho - \rho o\ \ /\ \rho o \quad \Rightarrow \partial\omega/\partial z$



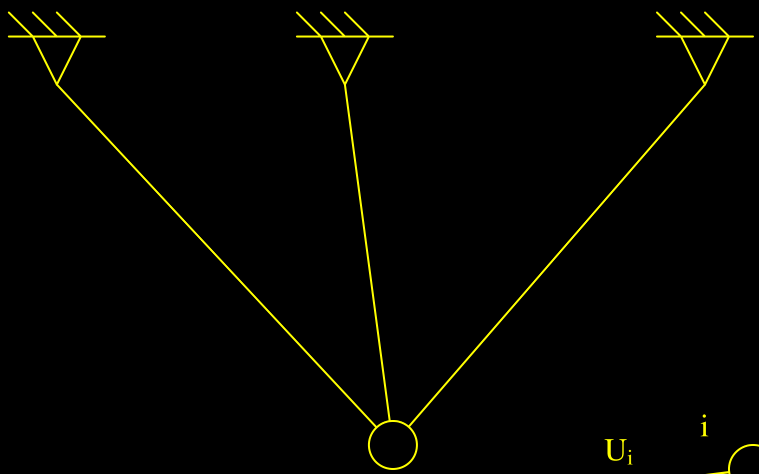
Poutre : toutes les solicitations

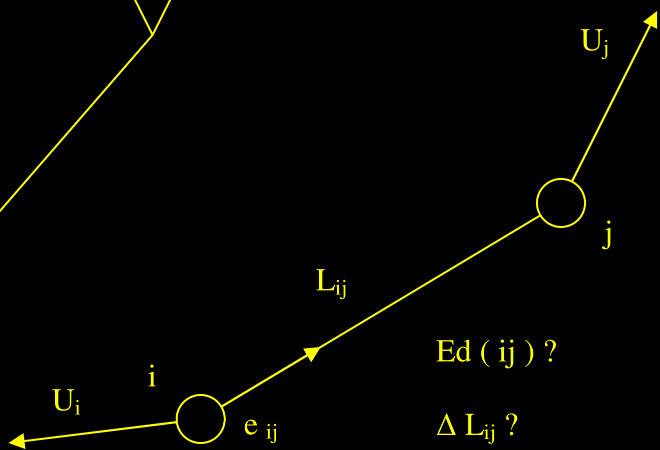
Barre traction/ compression

élargir sur



1 Analyse du treillis $E_D = f \quad u_1 \; v_1$



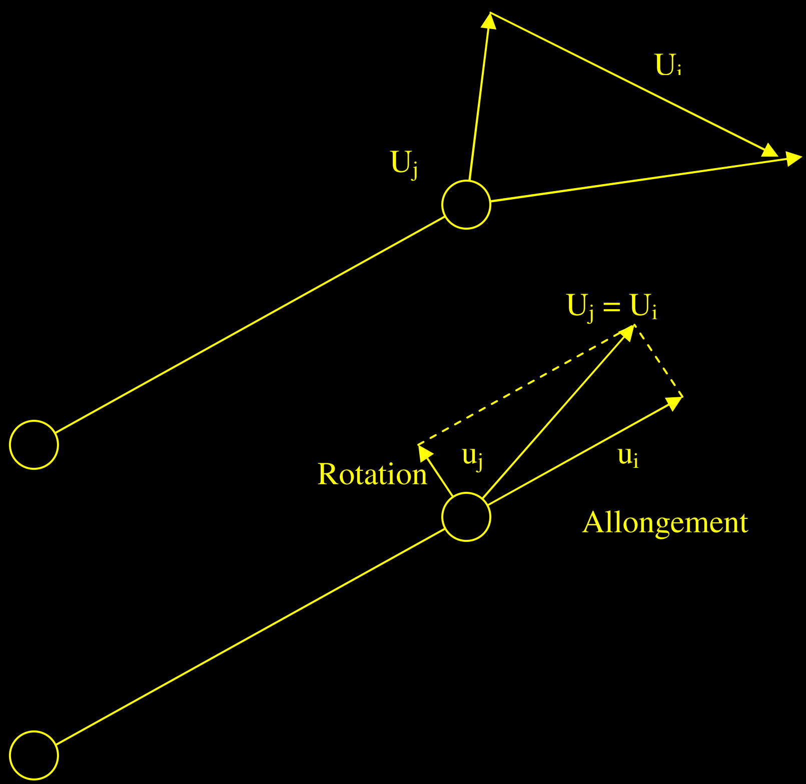


Ed (ij) ?

ΔL_{ij} ?

$\Delta L_{ij} = (U_j - U_i)\, e_{ij}$

Translation



Barre	U_j	U_i	e_{ij}	ΔL_{ij}	$\Delta L_{ij} / L_{ij}$	ED (ij)
(1 2) i j	0	u_1 x $-v_i$ y	y	$-v_1$	$-v_1/\rho$	$1/2\ ES/L\ v_1{}^2$
(1 3)	0	u_1 x $+v_1$ y	$x + y / \sqrt{2}$	$-u_1 - v_1/\sqrt{2}$	$-u_1 - v_1/2l$	$1/2(ES/\sqrt{2}\ l)/(u_1-v_1)/2$
(1 4)	0	u_1 x $+v_1$ y	$-x + y / \sqrt{2}$	$u_1 - v_1 / \sqrt{2}$	$u_1 - v_1/2l$	$1/2(ES/\sqrt{2}\ l)/(u_1-v_1)/2$

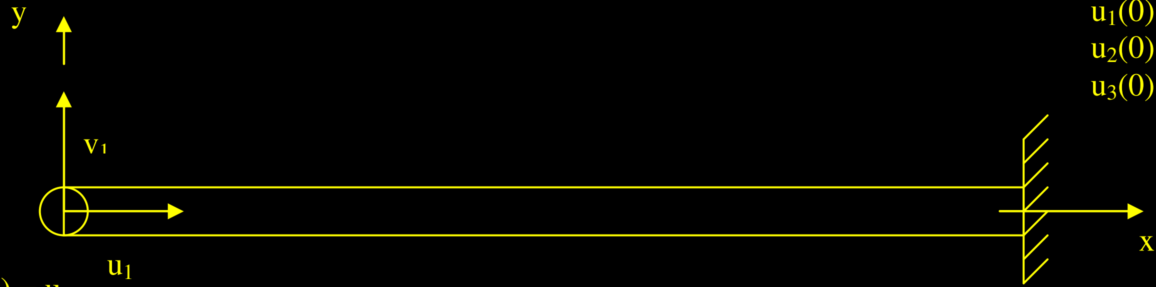
$E_d = 1/2 \ {}_0\int^{lxj} \ ES\varepsilon_1{}^2 \ dx$
$\qquad\qquad \Delta L_{ij} / L_{ij}$

A Analyse du treilli

$E_d = f^2\ u_1, v_1$

E_d (treillis) $= \Sigma\ E_d$ (barres
$= 1/2\ ES/\ 2\ \ v_1{}^2 + 1/2\ ES/2\sqrt{2}l\ \ [\ (u_1+v_1)^2 + (u_1-v_1)^2\ ]$
$= 1/2\ ES/\ 2\ v_1{}^2 + 1/2\ ES/\sqrt{2}l\ \ (u_1{}^2 + v_1{}^2)$

$u_1(0) = 0$
$u_2(0) = 0$
$u_3(0) = 0$



$u_1\ (0) = u_1$
$u_2(0) = v_1$
$Mf_3(0) = 0$

E_D (poutre) f ° de u_1, v_1
$E_D = 1/2\ {}_0\int^L \ (\ ES\ \varepsilon_1{}^2 + EI\ U_2''^2)\ dx$

Traction + Flexion sous Bernouilli

$U_1(x) = ax + b$
$U_2(x) = d°3 = Cx^3 + dx^2 + ex + f$
$dR/dx + 0 = 0 \quad \Rightarrow \quad N(x) = cste$
$T_2(x) = cste$

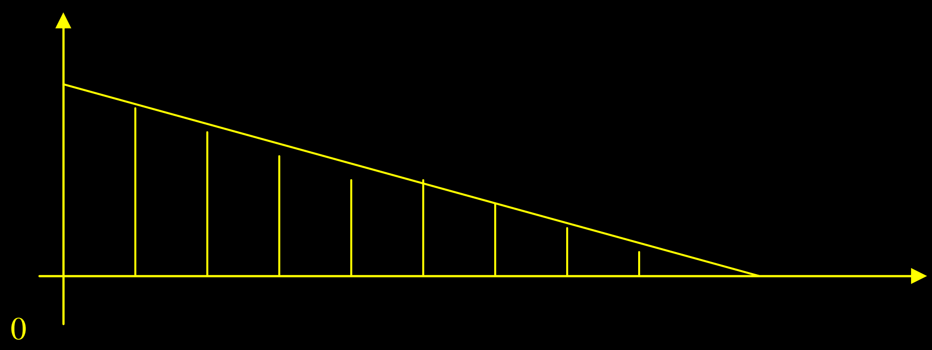
$N = ES\, u_1'$
$Mf_3 = EI\, u_2''$

$u_1(x)$ CA

$U_1(0) = U_1$ $\qquad$ $U_1(x) = U_1/L \; (L-x)$
$U_1(L) = 0$

$aL + b = 0$
$b = u_1$

$U_1(x) = U_1/L\,(L - x)$



$\varepsilon_1 = u_1(x) = -u_1 / L$

$u_2(x)$ CA

$u_2(x) = Cx^3 + dx^2 + ex + f$

$U_2(0) = v_1$
$U_1(L) = 0$
$U_2(L) = 0$

$f = v_1$
$CL^2 + dL^2 + e\,L + v_1 = 0$
$3CL^2 + 2dL + e = 0$

$f = v_2$
$Cl^3 + e_l + v_1 = 0$
$e = -3el^2$

reste une inconnue d

$\partial E_p/\partial p = 0$

$d=0$

Verif

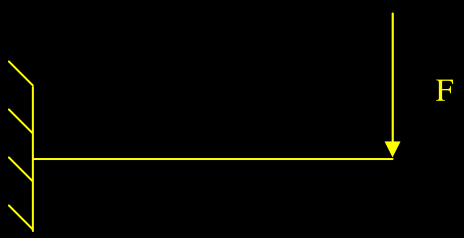
$U_2''(0) = 0$

$c = v_1/2l^3$ $\quad\quad\quad\quad$ $U_2\,(x) = v_1/2l^3\ x^3\ \ 3/2\ \ v_1/L\ \ x\ \ + v_1$

$e = -\ 3\ \ v_1\ /\ 2l$

$E_d\ (\text{poutre}) = 1/2\ \ {}_0\int L\ \ ES\,(u_1/L)^2 + EI\,(3v_1\ /\ L\ \ \ x\ )^2\,dx$

$= 1/2\ ES/L\ \ U_1{}^2\ + 1/2\ EI/L^2\ \ (\ 9v_1{}^2)\ \ L^3\ \ /\ 3$



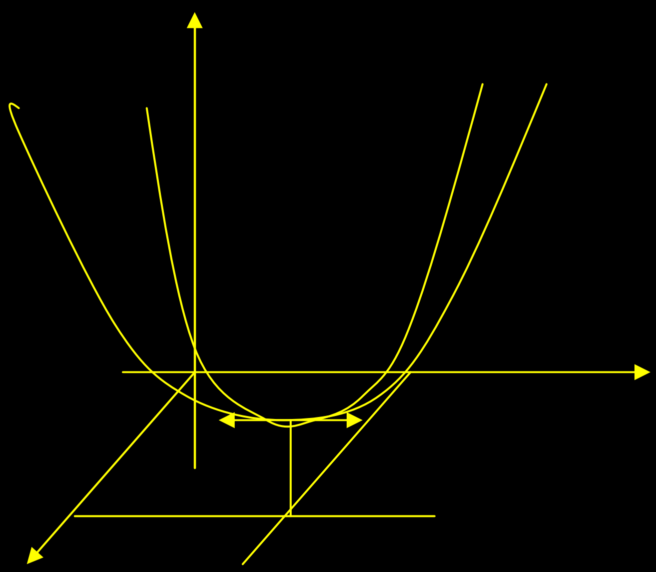
$1/2\ (\ 3EI\ /\ L^3\ )\ v_1{}^2$

B Structure Entière

$E_p = E_d\ (\text{treillis})\ \ + E_d\ \ (\ \text{poutre}\ )\ \ -\ \ W_{\ ext}$

$E_p\ (u_1\ ,\ v_1) = 1/2\ –ES/\rho\ \ v_1{}^2 + 1/2\ ES/\ v_2l\ (u_1{}^2{+}v_1{}^2) + 1/2\ ES/L\ u_1{}^2\ \ + 1/2\ 3EI/L^3\ \ v_1{}^2\ \ -\ P\ v_1$

Solution => Ed minimale



$\partial\ Ep/\partial u_1 = 0$

$\partial\ Ep\ /\ \partial v_1 = 1$

association parallele de resorts

$U_1 = 0$
$V_1 = P / ((ES/\rho + ES/(\sqrt{2}\ \ l)) + 3EI/L^3)$

$\partial E_d / \partial u_1 = \partial W_{ext} / \partial u_1$

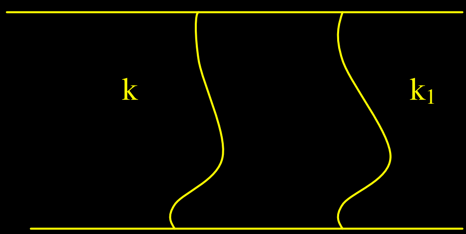
$\partial E_d / \partial v_1 = \partial W_{ext} / \partial v_1$

$$\begin{vmatrix} (ES/\sqrt{2}\ \ l) + ES/L & 0 \\ 0 & (ES/\rho + ES/\sqrt{2}\,l) + 3EI/L^3 \end{vmatrix} \begin{vmatrix} u_1 \\ v_1 \end{vmatrix} = \begin{vmatrix} 0 \\ P \end{vmatrix}$$

Remarque => $f = Py + Qx$
=> Si on retire (13)

$[\ \ ]\ \begin{matrix} u_1 \\ v_1 \end{matrix} = \begin{matrix} Q \\ P \end{matrix}$

$F = F_1 + F_2$
$= k_1\ \Delta l + k_2\ \Delta l$
$= (k_1 + k_2)$



Contrainte maxi

Treillis

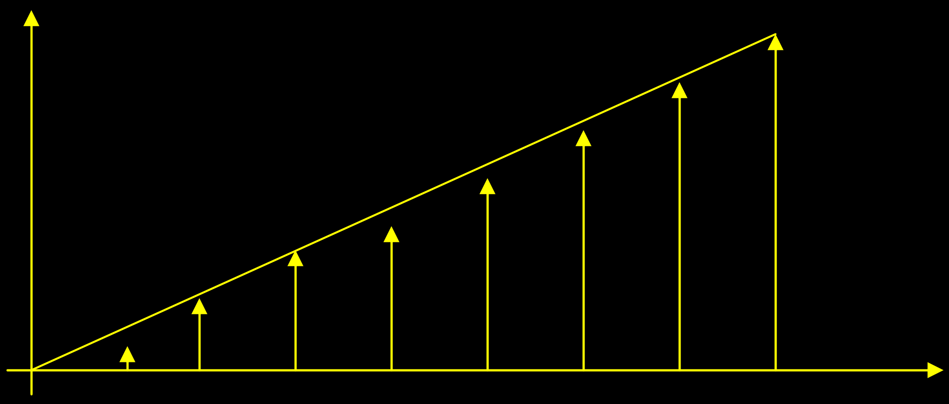
$U_1 = 0$

ε_1 (12) $= -v_1 / \rho$ => le plus constraint
ε_1 (13) $= -v_1/ 2\rho$

barre (12) $\sigma_{max} = \varepsilon\ v_i / \rho$

* Poutre ?

$EI\, U_2'' = 3\, v_1 / L^3 \quad X \quad e_i \;=\; r_f$



en $x = L$

$m \;\; f_{max} = 3EI\, v_1 / L^2$

Poutre

$\sigma_{max} = M_{f\,max} / I/h$
$= (\, 3EI\, v_1 / L^2)/\, (I / h)$
$= 3\, Eh / L^2 \;\; * \;\; v_1$

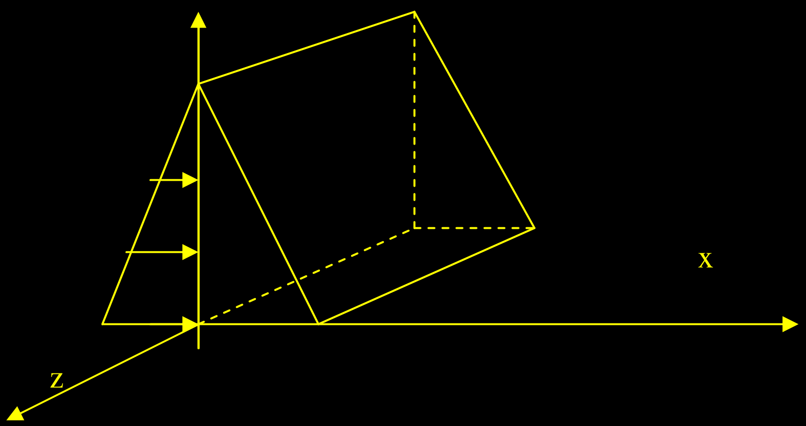
$\sigma_{max} / \sigma_{max}(12) \;\; = (\, 3\, Eh/ L^2 \;*\; v_1\,) \;\; / (\, E\, v_1 / \rho\,)$
$= \;3\, hl / L^2 \quad <<<< 1$

$L \sim= 1$
$h \;<<< L$

barre (12) le + sollicité

$\sigma_{12} < \; R_e / s$

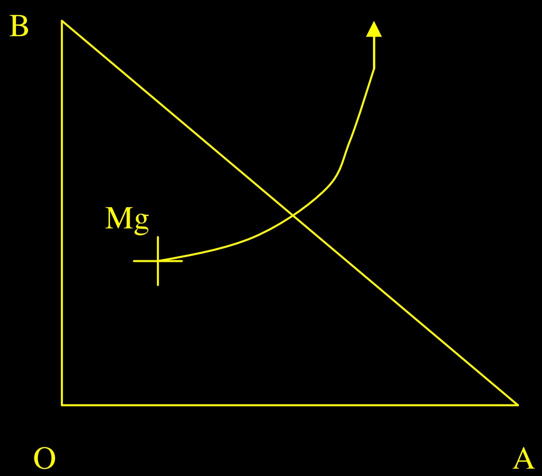
$E/\rho\;(\,Pcritique / + \;+\,)\;=\;R_e / s$
v_1



$$\varepsilon = \begin{pmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

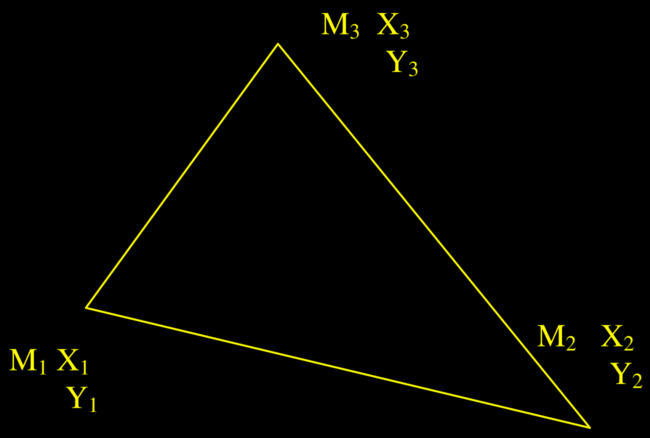
$U = u\,(x,y)\, e_x + v(x,y)\; e_y$

Révolution à 1 élément



Exprimer $U(u,v)$ en fonction de U_0, U_1, U_B

Triangle quelconque



aire $\neq 0$ M_1, M_2, M_3

non alignés

$U(u,v) = N_1(x,y)\, U(M_1) + N_2(x,y)\, U(M_2) + N_3(x,y)\, U(M_3)$

$U\,.\,(x_1, y_2) = U(M_1) \Rightarrow N_1(x_1, y_1) = 1$

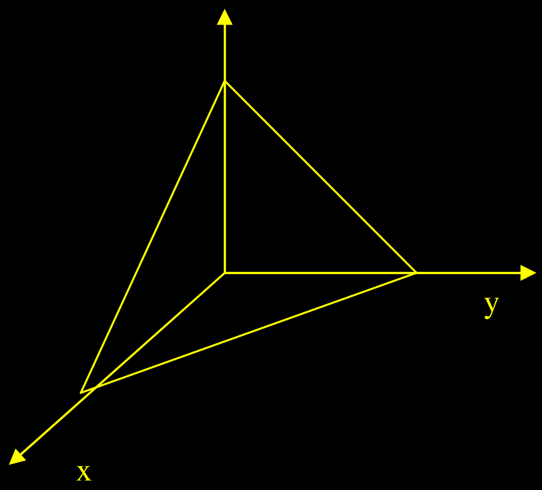
$N_1(x_2, y_2) = 0$

$N_1(x_3, y_3) = 0$

$ax_1 + by_1 + c = 1$

$ax_2 + by_2 + c = 0$

$ax_3 + by_3 + c = 0$



$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\det S = \begin{matrix} X_1 & Y_1 & 1 \\ X_2 - X_1 & Y_2 - Y_1 & 0 \\ X_3 - X_1 & Y_3 - Y_1 & 0 \end{matrix} = (X_2 - X)(Y_3 - Y_1) - (X_3 - X_1)(Y_2 - Y_1)$$

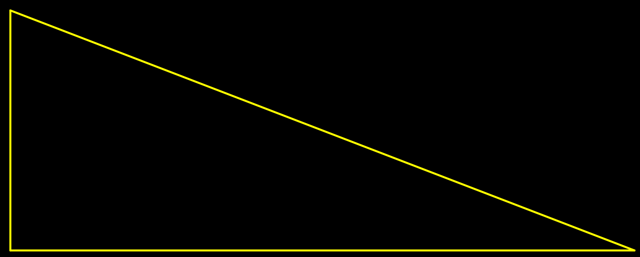
$$a = \frac{\begin{matrix} 1 & Y_1 & 1 \\ 0 & y_2 & 1 \\ 0 & y_3 & 1 \end{matrix}}{\det S} = \frac{Y_2 - Y_3}{\det S}$$

$$b = \frac{\begin{matrix} X_1 & 1 & 1 \\ X_2 & 0 & 1 \\ X_3 & 0 & 1 \end{matrix}}{\det S} = \frac{X_3 - X_2}{\det S}$$

$$c = \frac{\begin{matrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 0 \\ X_3 & Y_3 & 0 \end{matrix}}{\det S} = \frac{X_2 Y_3 - X_3 Y_2}{\det S}$$

Application

$X_3 = 0$
$Y_3 = 2L$



$X_1 = 2L$
$Y_1 = 0$

$X_2 = 0$
$Y_2 = 0$

$$\det S = (-2L)(2L) = -4L^2$$

$$N_1(x,y) = (6\ 2L/ -4l^2)\ x + (0/ L^2)\ y + 0 = x/ 2L$$

$$U(2) = (x/L)\ U() + ((L - x - y)/2L)\ U(2) + (y / 2L)\ U(3)$$

$$ED = 1/2\ (U_1\ \ U_2\ \ U_3\ \ V_1\ \ V_2\ \ V_3)\ [K°] \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{array}$$

$$Ed = 1/2\ (1) \iint tr\ (\varepsilon\ K\ \varepsilon)\ d\ \Omega$$

$$U \Rightarrow \varepsilon \Rightarrow \sigma = K\ \varepsilon \Rightarrow Ed$$

$$\begin{array}{cccccc} x/2L & 2L\text{-}x\text{-}y\ /2L & y & 0 & 0 & 0 \\ 0 & 0 & 0 & x/2L & N_2 & N_3 \end{array}$$

$$U = \begin{array}{cccccc} x/2L & 2l - x\text{-}y\ /\ 2L & y/\ L & 0 & 0 & 0 \\ 0 & 0 & 0 & x/2L & N_2 & N_3 \end{array} \quad \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{array}$$

$\varepsilon_{xx} = \partial u/\partial x = (1/2L)\, U_1 - (1/2L)\, U_2$
$\varepsilon_{yy} = \partial v/\partial y = (-1/2L)\, V_2 + (1/2L) V_3$
$\varepsilon_{xy} = 1/2(\partial u/\partial y + \partial v/\partial x)$
$= 1/2\, [\, (-1/L)\, U_2 + (1/2L) U_3 + (1/2L) V_1 - (1/2L) U_2]$

$$\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \end{array} = 1/2L \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 1/2 & -1/2 & 0 \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{array}$$

Relation de comportement / cas de déformation plane

$E\varepsilon = (1+\nu)\sigma - \gamma\, (\mathrm{tr}\, \sigma)\, 1$

$\varepsilon_{zz} = 0$
$= (1+\nu)\, \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$

$\sigma_{zz} = \nu(\, \sigma_{xx} + \sigma_{yy})$

$\mathrm{tr}\, \sigma = \sigma_{xx} + \sigma_{yy} + \gamma\, (\, \sigma_{xx} + \sigma_{yy}\,)$
$= (1 + \nu)\, (\sigma_{xx} + \sigma_{yy})$

$E\varepsilon_{xx} = (1+\nu)\, \sigma_{xx} - \nu(1+\nu)(\sigma_{xx}+\sigma_{yy})$
$E\varepsilon_{yy} = (1+\nu)\, \sigma_{yy} - \nu(1+\nu)(\sigma_{xx}+\sigma_{yy})$
$E\varepsilon_{xy} = (1+\nu)\, \sigma_{xy}$

relation σ.ε en DP

$$\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yx} \end{array} = E/(1+\nu)(1-2\nu) \begin{array}{cccc} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 1-2\nu \end{array} \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \end{array}$$

$E_d = 1/2 \; {}_{el}t\iint \mathrm{tr}\, (\sigma\varepsilon)\, ds$ ©
$= 1/2 \iint \varepsilon^{T} \sigma\, ds$
$= 1/2\, (2L^2/2)\, \varepsilon^{T} [C]\, \varepsilon$
$= 1/2\, (2L^2)\, U^{T} [D]^{T} [C][D]\, U$
$= 1/2\, U^{T} [(2L^2)[D]^{T} [C][D]U$

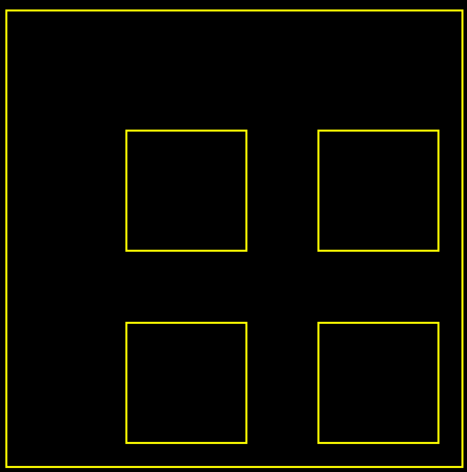
$$[K] = (2L^2)(1/2L)^2 \begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & 0 & -1/2 & -1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & -1 & -1/2 & 1/2 \\ 0 & 1 & 0 & 0 \end{array} \; E/(1+\gamma)(1-2\nu) \begin{array}{cccc} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 1-2\nu \end{array}$$

$$W_{ext} = (1)\ {}_0\int^{2L} (\rho y)(x-y)_e x)\ (2L-y\ /2L\ \ u_2\ e_x + y/2L\ U_3\ e_x)\ dy$$
$$= (\ U_2\ U_3)\ \begin{matrix} 4/3\rho g L^2 \\ 2/3\rho g L^2 \end{matrix}$$

$$\partial E_d/\partial U_i = \partial W_{ext}/\ \partial u_i$$

$$E_d = 1/2\ ((2/L)^2 / 2\ )\ \varepsilon^T\ [C]\ \varepsilon$$
$$= 1/2\ U^T\ [\ (2\ L^2)\ [D]^T\ [C]\ [D]]\ U$$

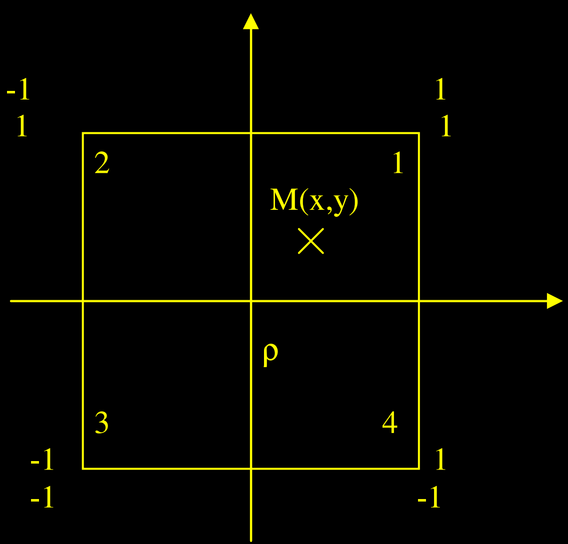
$$[K]\ U = F$$

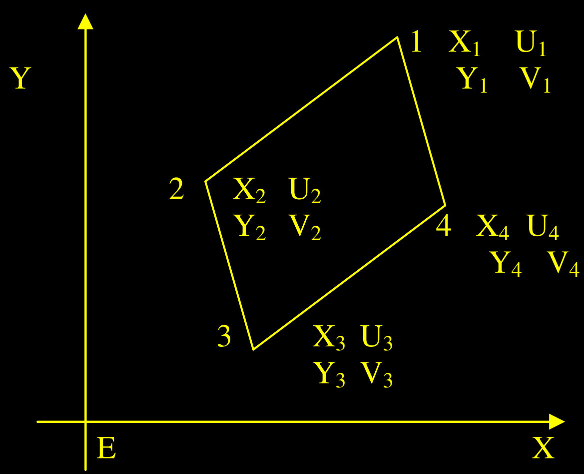


$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ F_2 \\ F_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad 1/2\ (\ (2L)^2/2)\ \varepsilon^T\ [C]\ \varepsilon$$

$$Ed = 1/2\ (2L^2)\ U^T\ [D]\ ^T\ [C]\ [D]$$
$$=\ U^T\ [(2L^2)\ [D]^T\ [D][D]]\ U$$

2 Quadrilatère à 4 nœuds





$$U_M \quad \begin{matrix} U \\ V \end{matrix} = \begin{matrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{matrix} \quad \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix}$$

$N_1 (x,y) = \ 1/4 (x+1)(y+1)$
$N_2 (x,y) = -1/2 (x-1)(y+1)$
$N_3 (x,y) = \ 1/4 (x-1)(y-1)$
$N_4 (x,y) = -1/4 (x+1)(y-1)$

De même pour les coordonnées

$X(x,y) = X_1N_1(x,y) + X_2N_2 + X_3N_3 + X_4N_4$
$y(x,y) = Y_1N_1 + Y_2N_2 + Y_3N_3 + Y_4N_4$

$$\begin{matrix} N^T & O^T & X \\ O^T & N^T & Y \end{matrix}$$

MEF => E_p (U)

Objectif

$E_d = 1/2 \ E\int Tr [\sigma (\varepsilon) \varepsilon] dE$
$\quad = 1/2 \ e\int Tr[\sigma(\varepsilon)\varepsilon] J \, dx \, dy$

Intégration Numérique

$E_d = 1/2 [U^T, V^T] K^{el} [U^T V^T]$

Intégration aux points de Gauss
=>Evaluée au point
calcul du poids
=>point par point
intégrale équivalente

Calcul de J (Jacobien)

$$F = \begin{matrix} \partial X/\partial x & \partial X/\partial y \\ \partial Y/\partial x & \partial Y/\partial Y \end{matrix}$$

avec $\partial X/\partial x = 1/4\,[X_1(y+1) - X_2(y+1) + X_3(y-1) - X_4(y-1)]$
$\quad\quad = 1/4\,[(X_2 - X_2(y+2) - (X_3 - X_4)(y-1)]$

et $\quad \partial Y/\partial y = 1/4\,[Y_1(x+1) + Y_2\,(x-1) - Y_3\,(x-1) - Y_4\,(x+1)]$
$\quad\quad = 1/4\,[\,(Y_1 - Y_4)(x+1) + (Y_2 - Y_3)(x-1)]$

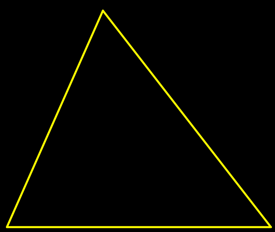
$\partial X/\partial y = 0 \quad \partial Y/\partial x = 0$

$J = \det F$
$\quad = [\,\partial X/\partial x\ \ \partial Y/\partial y - \partial X/\partial y\ \ \partial Y/\partial x\,]$

$J = 1/16\ \ [[\,(X_1 - X_2)(y+1) + (X_3 - X_4)(y-1)] * [(Y_1 - Y_4)\,(x+1) + (Y_2 - Y_3)(x-1)\,]]$

$J = 1/16[[\,(X_1 - X_2)(y+1) + (3 - X_1)(y-1)][(Y_1 - Y_4)(x+1) + (Y_2 - Y_3)(x-1)]]$

[pour $\quad\quad\quad\quad J = 2$ Aire (E)]



$$\varepsilon(u) = \begin{matrix} \partial U/\partial x & 1/2\,(\partial U/\partial Y + \partial V/\partial X) \\ 1/2(\,\partial U/\partial y + \partial V/\partial x) & \partial V/\partial y \end{matrix}$$

$$\varepsilon = \begin{matrix} \varepsilon xx \\ \varepsilon yy \\ \varepsilon xy \\ \varepsilon yx \end{matrix} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{matrix} \begin{matrix} \partial U/\partial x \\ \partial U/\partial y \\ \partial V/\partial x \\ \partial V/\partial y \end{matrix}$$

$$\begin{matrix} \partial U(x,y)\,/\,\partial X & = \partial u \partial z\,/\,\partial x \partial X & + & \partial U \partial y/\partial y \partial X & = & \partial x/\partial X & \partial y/\partial Y & \partial U/\partial x \\ \partial U(x,y)\,/\,\partial Y & \partial u \partial x\,/\,\partial x \partial Y & + & \partial U \partial y/\partial y \partial Y & = & \partial x/\partial Y & \partial y/\partial Y & \partial U/\partial y \end{matrix}$$

$$F = \begin{matrix} \partial X/\partial x & \partial X/\partial y \\ \partial Y/\partial x & \partial Y/\partial y \end{matrix} \quad \Rightarrow \quad F^{-1} = 1\,/\,\det F \ \begin{matrix} \partial Y/\partial y & -\,\partial X/\partial y \\ -\partial X/\partial x & \partial X/\partial x \end{matrix}$$

$$\begin{matrix} \partial U/\partial X \\ \partial U/\partial Y \\ \partial V/\partial X \\ \partial V/\partial Y \end{matrix} = \begin{matrix} F^{-1} & 0 \\ 0 & (F^{-1})^T \end{matrix} \begin{matrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{matrix}$$

$$\varepsilon = \begin{matrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\\ \varepsilon_{xy}\end{matrix} = \begin{matrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1/2 & 1/2 & 0\\ 0 & 1/2 & 1/2 & 0\end{matrix} \quad \begin{matrix}F^{-1} & 0\\ 0 & (F^{-1})^T\end{matrix} \quad \begin{matrix}N_{,x}{}^T & O^T\\ N_{,y}T & O^T\\ O^T & N_{,x}{}^T\\ O^T & N_{,y}{}^T\end{matrix} \quad \begin{matrix}U\\ V\end{matrix}$$

On a
$$\begin{matrix}U\\ V\end{matrix} \quad \begin{matrix}N^T & O^T\\ O^T & N^T\end{matrix} \quad \begin{matrix}U_1\\ U_2\\ U_3\\ U_4\\ V_1\\ V_2\\ V_3\\ V_4\end{matrix}$$

$$\partial u/\partial x = N_{,x}{}^T \; U$$

Finalement

$$\varepsilon = B \quad \begin{matrix}U\\ V\end{matrix}$$

avec
$$B = \begin{matrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1/2 & 1/2 & 0\\ 0 & 1/2 & 1/2 & 0\end{matrix} \quad \begin{matrix}F^{-1} & 0\\ 0 & (F^{-1})^T\end{matrix} \quad \begin{matrix}N_{,x}{}^T & O^T\\ N_{,y}{}^T & O^T\\ O^T & N_{,x}{}^T\\ O^T & N_{,y}{}^T\end{matrix}$$

$$\sigma = \begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\\ \sigma_{yx}\end{matrix} = K^{el} \; \varepsilon$$

$$E_d = 1/2 \; {}_E\!\int \; Tr\,[\,\sigma\,(\,\varepsilon\,)\,\varepsilon\,]\,dE \qquad \Rightarrow \qquad \sigma\varepsilon \;=\; \sigma^T\varepsilon$$

$$= 1/2 \; {}_E\!\int \; \varepsilon^T \;\; K^{\hat{e}l} \;\; \varepsilon \quad dE$$

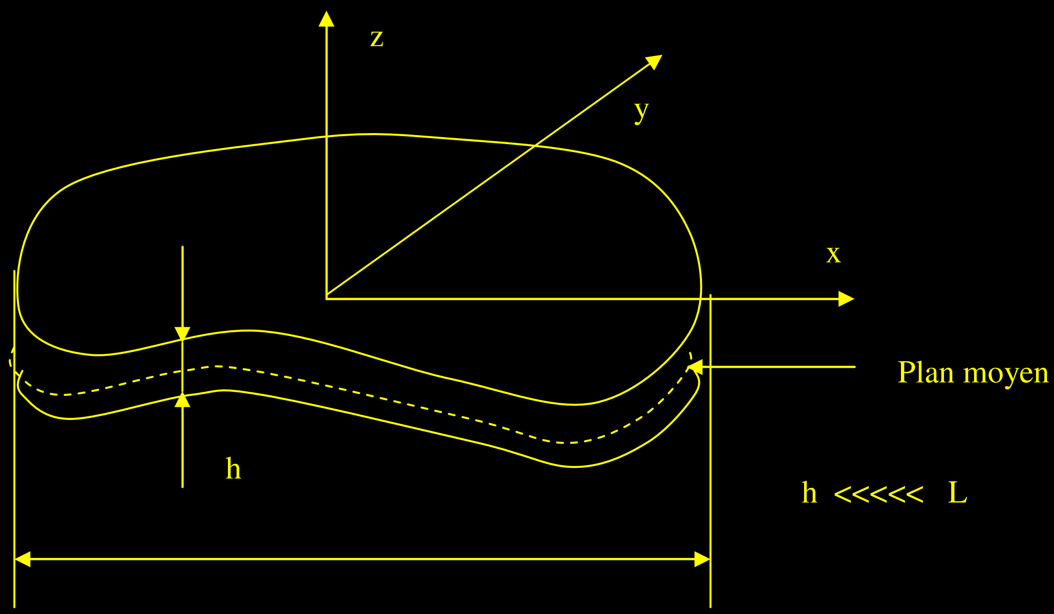
$$E_d = 1/2 \quad (e[U^T,V^T]B^T)\int \; \varepsilon^T \quad K^{el} \;\; \varepsilon \;\; J \;\; dx \;\; dy$$

$$E_d \; = 1/2 \;\; [U^T,V^T] \;\; e \int \; \underbrace{B^T \, K^{EP} \;\; B \; J \; dx \; dy}_{K^{el} \; !!!} \;\; [U \; V]$$

intégration numérique

Théorie des plaques ~= Théorie des poutres

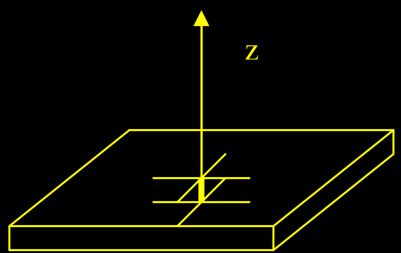
1 Généralités

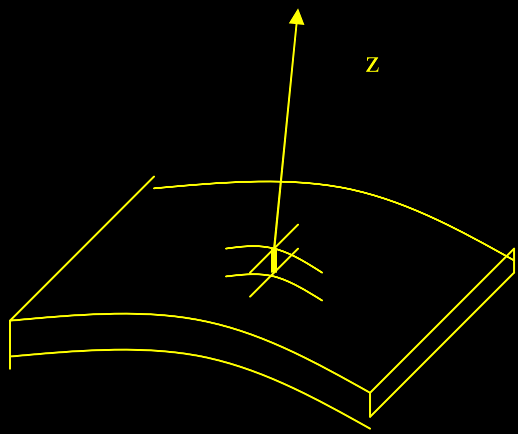


Hypothèse Cinématique: (Champ virtuel pour le th P.V.)

a) Les normales au plan moyen initial restent droites

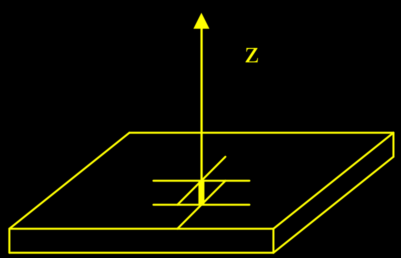
=> Théorie de Reissner Mindlin

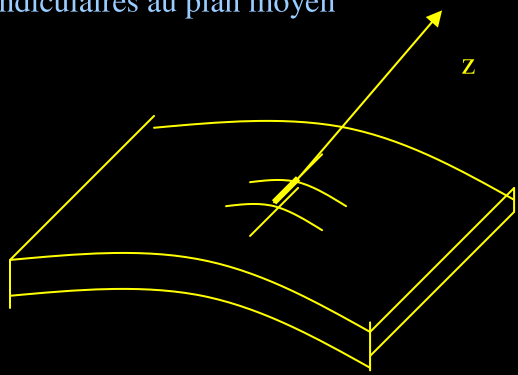




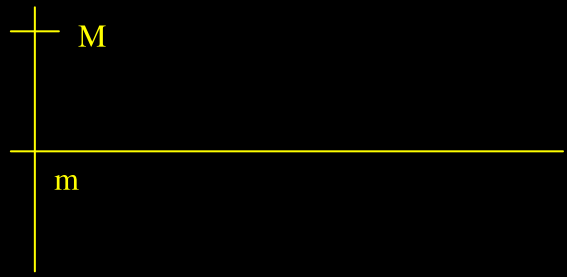
b) Théorie de Kirchhoff –Love-

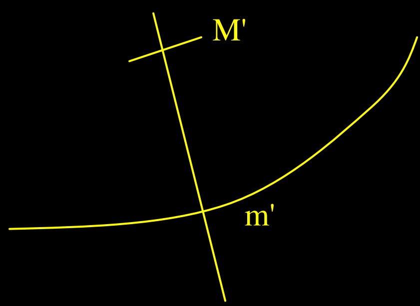
Les normales au plan moyen initial restent droites et perpendiculaires au plan moyen





Déplacement initial





Déplacement des corps rigides

$$U_M^* \,(x,y,z) = U_m(x,y)^* + \omega_m(x,y)^* \wedge mM$$

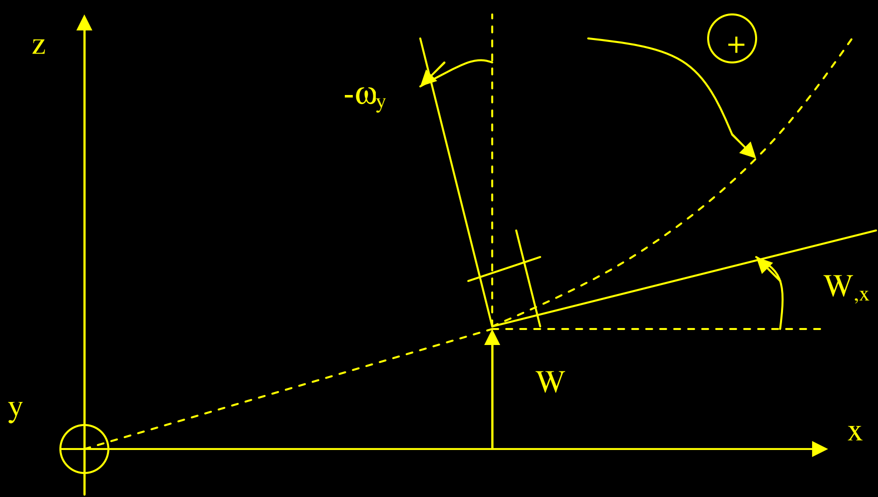
$$U_M^* = \begin{matrix} U^*(x,y) \\ V^*(x,y) \\ W^*(x,y) \end{matrix} + \begin{matrix} \omega_x^*(x,y) \\ \omega_y^*(x,y) \\ \omega_z^*(x,y) \end{matrix} \wedge \begin{matrix} 0 \\ 0 \\ Z \end{matrix} = \begin{matrix} U^* + \omega_y^* \;\; y \\ V^* - \omega_x^* \;\; z \\ W^* \end{matrix}$$

$$\begin{aligned}
\varepsilon^* \; (U_M^*) \quad \text{avec} \quad \varepsilon_{xx}^* &= U_{x,x}^* \\
&= U_{,x}^* + \omega_{y,x}^* \; z \\
\varepsilon_{xy}^* &= 1/2\,(U_{x,y}^* + U_{y,x}^*) \\
&= 1/2\,(U_{,y}^* + V_{,x}^* + \omega_{y,y}^* \; z - \omega_{x,x}^* \; z) \\
\varepsilon_{xy}^* &= 1/2\,(U_{x,z}^* + U_{z,x}^*) \\
&= 1/2\,(\omega_y^* + W_{,x}^*) \\
\varepsilon_{yy}^* &= U_{y,y}^* \\
&= V_{,y}^* - \omega_{x,y}^* \; z \\
\varepsilon_{yz}^* &= 1/2\,(U_{y,z}^* + U_{z,y}^*) \\
&= 1/2\,(-\omega_x^* + W_{,y}^*) \\
\varepsilon_{zz}^* &= U_{z,z}^* \\
&= 0
\end{aligned}$$

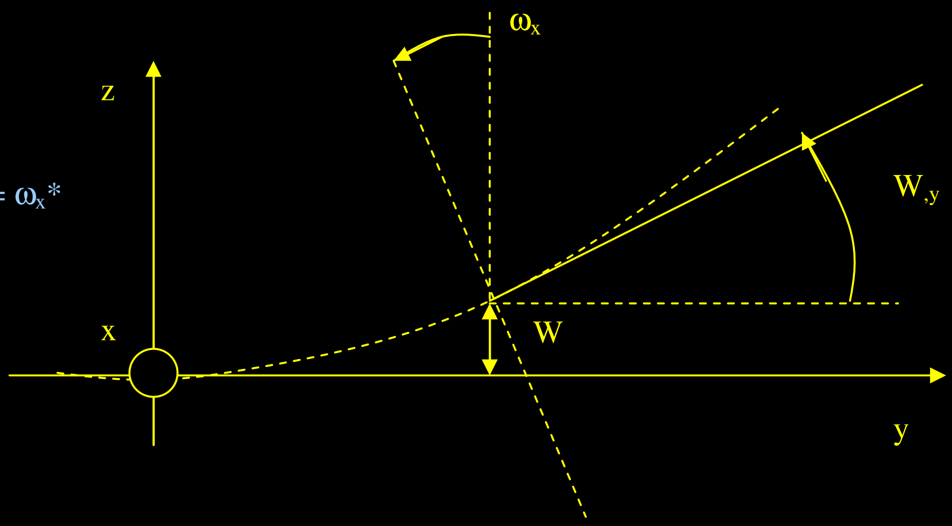
$$\varepsilon^* \,(U_M^*) = \begin{matrix} \varepsilon_{xx}^* & \varepsilon_{xy}^* & \varepsilon_{xz}^* \\ \varepsilon_{xy}^* & \varepsilon_{yy}^* & \varepsilon_{yz}^* \\ \varepsilon_{xz}^* & \varepsilon_{zy}^* & \varepsilon_{zz}^* \end{matrix}$$

Si les segments sont orthogonaux au plan moyen (Krisschoff – love)

$\Rightarrow \; W_{,x}^* = -\omega_y^*$



$\Rightarrow W_{,y}^* = \omega_x^*$



$$\varepsilon^* = \begin{bmatrix} U_{,x}^* - W_{,xx}^* & \frac{1}{2}(U_{,y}^* + V_{,x}^*) - W_{,xy}^* \, z & 0 \\ \frac{1}{2}(U_{,y}^* + V_{,x}^*) - W_{,xy}^* \, z & V_{,y}^* - W_{,yy}^* \, z & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Théorème des travaux virtuels = puissance

$$P_{int}^* + P_{ext}^* = 0 \; \forall \; U^*$$

$$P_{int}^* = - {}_{\Omega}\!\int \mathrm{Tr}\,[\,\sigma\, \varepsilon^*\,]\, d\Omega$$

$$P_{int}^* = - \int_{(-h/2)}^{(h/2)} \int_{Sm} (\sigma_{xx}\, \varepsilon_{xx}^* \;\; \sigma_{yy}\, \varepsilon_{yy}^* + 2\sigma_{xy}\, \varepsilon_{xy}^*)\, dS\, dz$$

$$P_{int}^* = - \int_{Sm} \int_{(-h/2)}^{(h/2)} \sigma_{xx}\,(U_{,x}^* - W_{,xx}^*\, z) + \sigma_{yy}(V_{,y}^* - W_{,yy}^*\, z) + \sigma_{xy}(U_{,y}^* + V_{,x}^* - 2W_{,xy}^*\, z)\, dz\, dS$$

$$P_{int}^* = - \int_{Sm} U_{,x}^* \int_{(-h/2)}^{(h/2)} \sigma_{xx}\, dz + V_{,y}^* \int_{(-h/2)}^{(h/2)} \sigma_{yy}\, dz + U_{,y}^* + V_{,x}^* \int_{(-h/2)}^{(h/2)} \sigma_{xy}\, dz - W_{,xy}^* \int_{(-h/2)}^{(h/2)} \sigma_{xx}\, z\, dz - W_{,yy}^* \int_{(-h/2)}^{(h/2)} \sigma_{yy}\, z\, dz - 2W_{,xy}^* \int_{(-h/2)}^{(h/2)} \sigma_{xy}\, z\, dz$$

Puissance virtuelle des efforts interieurs

$$P_{int}^* = - \int_{Sm} (\, N_{xx}\, \varepsilon_{oxx}^* + N_{yy}\, \varepsilon_{oxy}^* + 2N_{xy}\, \varepsilon_{oxy}^* + M_{xx}\, \gamma_{xx}^* + M_{yy}\, \gamma_{yy}^* \;\; 2M_{xy}\, \gamma_{xy}^*)\, dS$$

$$= - \int_{Sm} \left[\mathrm{Tr}\, [N \;\; \varepsilon_o^*] \;\; + \;\; \mathrm{Tr}\, [\; M \;\; \gamma^* \;] \right] dS$$

$$\begin{bmatrix} N_{xx} & N_{xy} \\ N_{xy} & N_{yy} \end{bmatrix} \qquad \begin{bmatrix} U_{,x}^* & \frac{1}{2}(U_{,y}^* + V_{,x}^*) \\ \frac{1}{2}(U_{,y}^* + V_{,x}^*) & V_{,y}^* \end{bmatrix} \qquad \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} \qquad \begin{bmatrix} -W_{,xx}^* & -W_{,xy}^* \\ -W_{,xy}^* & -W_{,yy}^* \end{bmatrix}$$

Effort de tension — Contraintes planes — Moment — Courbures

T P V

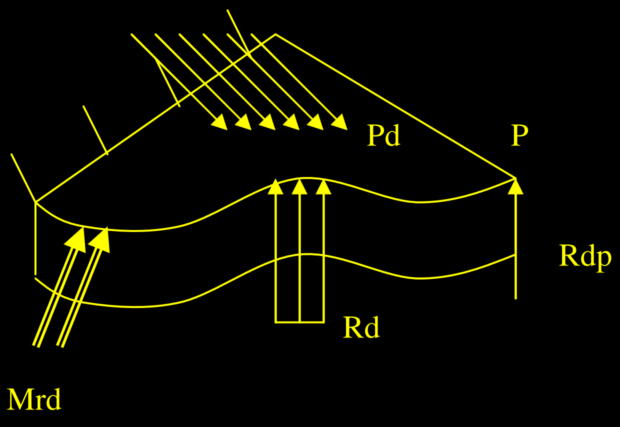
$P_{int}* + P_{ext}*$

$\mathrm{div}\,(\mathrm{div}\,M) + P_d = 0$ Sur S

n M n = Mrd sur ∂2 Sm

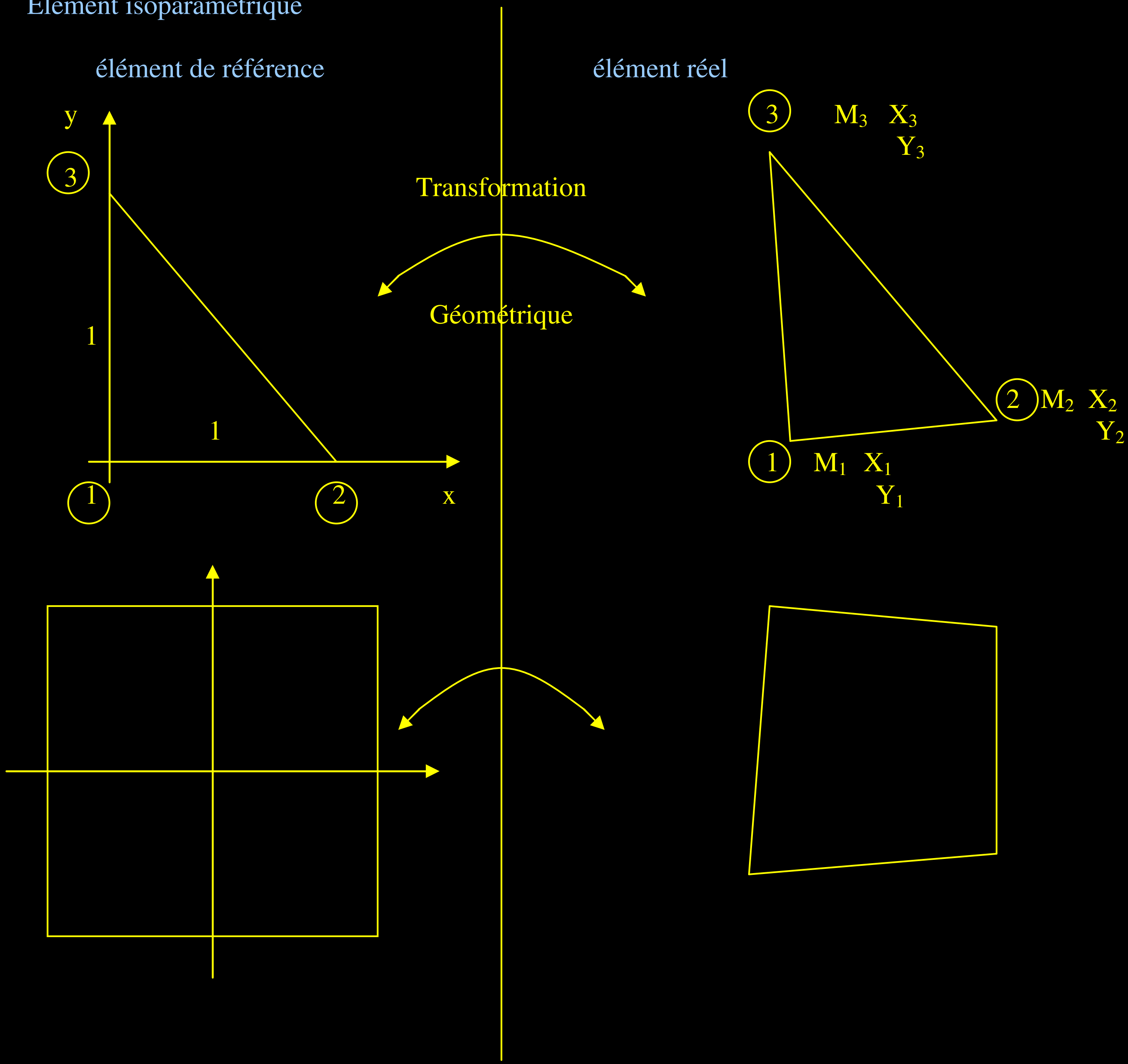
div (Mn) + (t. Mn),s = Rd

[t. Mn] p = R dp

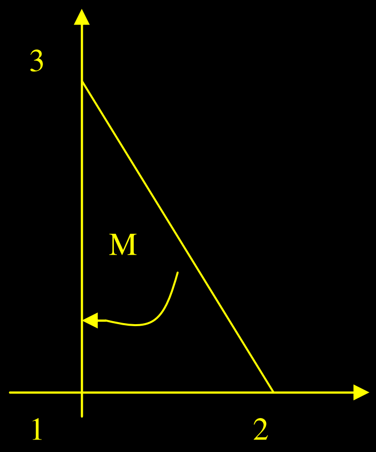


Calcul de structure

Elément isoparamétrique



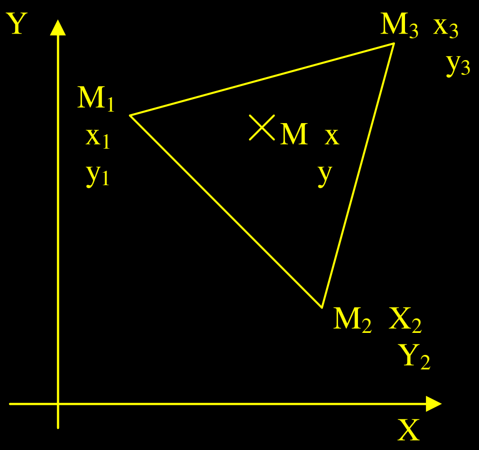
ISO



$$U = \begin{matrix} 1-x-y & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-x-y & x & y \end{matrix} \quad \begin{matrix} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{matrix}$$

Les fonctions d'interpolation permettant de definir le déplacement en tout point de l'élément iso

REEL



Position de $M \in E^*$

$$\begin{matrix} X \\ \\ Y \end{matrix} = \begin{matrix} 1-x-y & x & y & 0 & 0 & 0 \\ \\ 0 & 0 & 0 & 1-x-y & x & y \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{matrix}$$

$0<x<1$ x,y parametres coordonne
$0<y<1$ des sommets

Ed (élément réel)

$$\varepsilon xx = \partial U/\partial X = \partial U/\partial x \, \partial x/\partial X + \partial U/\partial y \, \partial y/\partial X$$

$$\varepsilon yy = \partial U/\partial Y = \partial U/\partial x \, \partial x/\partial Y + \partial U/\partial y \, \partial y/\partial Y$$

$$\varepsilon xy = 1/2 \, (\partial U/\partial Y + \partial U/\partial X)$$

Transformation

(x,y) (X,Y)

$$F = \boxed{\begin{matrix} \partial X/\partial x & \partial X/\partial y \\ \partial Y/\partial x & \partial Y/\partial y \end{matrix}} = \boxed{\begin{matrix} -X_1 + X_2 & -X_1 + X_3 \\ -Y_1 + Y_2 & -Y_1 + Y_3 \end{matrix}}$$

F gradient de la fonction de transformation

$$\begin{matrix} \partial U/\partial X \\ \partial U/\partial Y \end{matrix} = \underbrace{\boxed{\begin{matrix} \partial x/\partial X & \partial y/\partial X \\ \partial x/\partial Y & \partial y/\partial Y \end{matrix}}}_{(F^T)^{-1}} \begin{matrix} \partial u/\partial x \\ \partial u/\partial y \end{matrix}$$

$$\begin{matrix} \partial u/\partial x \\ \partial u/\partial y \end{matrix} = \underbrace{\boxed{\begin{matrix} \partial X/\partial x & \partial Y/\partial x \\ \partial X/\partial y & \partial Y/\partial y \end{matrix}}}_{F^T} \begin{matrix} \partial U/\partial X \\ \partial U/\partial Y \end{matrix}$$

$X = (1\text{-}x\text{-}y)X_1 + xX_2 + yX_3$
$Y = (1\text{-}x\text{-}y)Y_1 + xY_2 + yY_3$

$X - X_1 = (X_2 - X_1)\, x + (X_3 - X_1)\, y$
$Y - Y_1 = (Y_2 - Y_1)x + (Y_3 - Y_1)\, y$

$\det S = (X_2 - X_1)(Y_3 - Y_1) - (X_3 - X_1)(Y_2 - Y_1)$

$$x = \frac{\begin{vmatrix} X - X_1 & X_2 - X_1 \\ Y - Y_1 & Y_2 - Y_1 \end{vmatrix}}{\det S}$$

$$= \frac{(X - X_1)(Y_2 - Y_1) - (Y - Y_1)(X_2 - X_1)}{(X_2 - X_1)(Y_3 - Y_1) - (Y_2 - Y_1)(X_3 - X_1)}$$

$$y = \frac{\begin{vmatrix} X - X_1 & X_3 - X_1 \\ Y - Y_1 & Y_3 - Y_1 \end{vmatrix}}{\det S}$$

$$= \frac{(X - X_1)(Y_3 - Y_1) - (Y - Y_1)(X_3 - X_1)}{(X_2 - X_1)(Y_3 - Y_1) - (Y_2 - Y_1)(X_3 - X_1)}$$

$$\partial x/\partial X = \frac{(Y_2 - Y_1)}{(X_2 - X_1)(Y_3 - Y_1) - (Y_2 - Y_1)(X_3 - X_1)}$$

$$\partial x/\partial Y = \frac{(X_2 - X_1)}{(X_2 - X_1)(Y_3 - Y_1) - (Y_2 - Y_1)(X_3 - X_1)}$$

$$\partial y/\partial X = \frac{(Y_3 - Y_1)}{(X_2 - X_1)(Y_3 - Y_1) - (Y_2 - Y_1)(X_3 - X_1)}$$

$$\partial y/\partial Y = \frac{(X_3 - X_1)}{(X_2 - X_1)(Y_3 - Y_1) - (Y_2 - Y_1)(X_3 - X_1)}$$

$$\begin{matrix} \partial U/\partial X \\ \partial U/\partial Y \\ \partial V/\partial X \\ \partial V/\partial Y \end{matrix} = \boxed{\begin{matrix} \boxed{(F^T)^{-1}} & 0 \\ 0 & (F^T)^{-1} \end{matrix}} \quad \begin{matrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{matrix}$$

$\varepsilon_{xx} = \partial\, u / \partial\, X$

$\varepsilon_{yy} = \partial\, v / \partial\, Y$

$\varepsilon_{xy} = 1/2\ (\partial u/\partial Y + \partial v/\partial X)$

$$\begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sqrt{2}\varepsilon_{xy} \end{matrix} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{matrix} \quad \begin{matrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{matrix}$$

$\sigma = \lambda\ (\mathrm{Tr}\ \varepsilon)\ 1 + 2\ \mu\ \varepsilon$

$$\begin{matrix} \sigma_{xx} \\ \sigma_{yy} \\ \sqrt{2}\sqrt{xy} \end{matrix} = \begin{matrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\,\mu & 0 \\ 0 & 0 & 2\mu \end{matrix} \quad \begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sqrt{2}\ \varepsilon_{xy} \end{matrix}$$

Cas déformations plannes

$$\begin{matrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{matrix} = \boxed{\begin{matrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{matrix}} \quad \begin{matrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{matrix}$$

$$
\begin{Bmatrix} u \\ v \end{Bmatrix} = \boxed{\begin{matrix} 1-x-y & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-x-y & x & y \end{matrix}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}
$$

$$E_d\,(2) = 1/2 \iint_{elt} \mathrm{tr}\,(\varepsilon \;\; K \;\; \varepsilon) \;\; d\Omega$$

$$\sigma_{xx}\,\varepsilon_{xx} + \sigma_{yy}\,\varepsilon_{yy} + 2\,\sigma_{xy}\,\varepsilon_{xy}$$

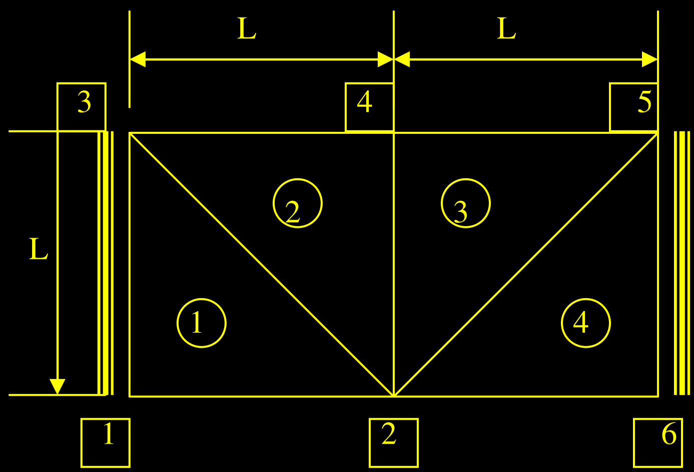
$$E_d\,(2) = 1/2 \iint \hat{\varepsilon}^T \; \hat{\sigma}\; dS$$

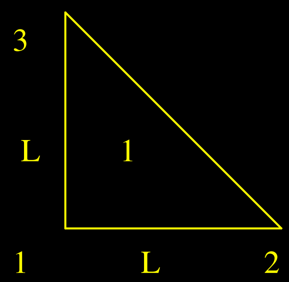
$$= 1/2 \iint \hat{\varepsilon}^T \; [\,C\,]\; \hat{\varepsilon}\; dS$$

$$= \; 1/2 \iint \hat{U}^\circ\; [B]^T[C][B]\; (\hat{U}^\circ)\; dS$$

$$E_d = 1/2 \iint_{elt} \hat{U}^{*T}\; [D^T]\; [B^T][C][B][D]\; (\hat{U}^*)\; dS$$

$$= 1/2\;(\text{Aire élément})\; U^T\; \underbrace{[A^T][D^T][B^T][C][B][D][A]}_{K^e}\; U$$

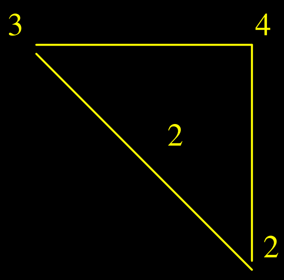




$$X_1 \;=\; 0 \qquad X_2 \;=\; 1 \qquad X_3 \;=\; 0$$

$$Y_1 \;=\; 0 \qquad Y_2 \;=\; 0 \qquad Y_3 \;=\; 1$$

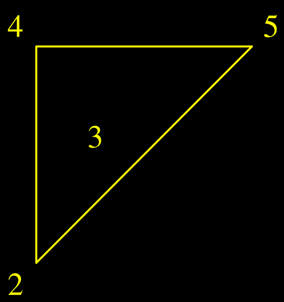
$$
[\,K\,(1)\,] \;=\; 1\,/\,2\; l^2 \quad \begin{array}{ccc} (\lambda + 2\,\mu)\; l^2 & 0 & U_2 \\ 0 & \mu\; l^2 & V_2 \end{array}
$$



$X_1 = 1 \quad X_2 = 1 \quad X_3 = 0$

$Y_1 = 0 \quad Y_2 = 1 \quad Y_3 = 1$

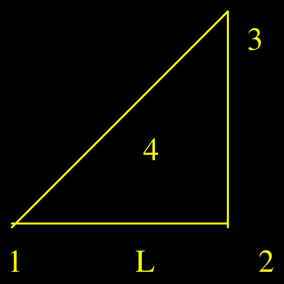
$$[K(2)] = 1/2 \quad \begin{array}{cccc|c} \mu & -\mu & 0 & \mu & U_2 \\ 0 & \lambda + 3\mu & -\lambda & \lambda+\mu & U_4 \\ 0 & 0 & \lambda + 2\mu & -(\lambda + 2\mu) & V_2 \\ 0 & 0 & 0 & \lambda + 3\mu & V_4 \end{array}$$



$X_1 = 1 \quad X_2 = 1 \quad X_3 = 21$

$Y_1 = 0 \quad Y_2 = 1 \quad Y_3 = 1$

$$[K(3)] = 1/2 \quad \begin{array}{cccc|c} \mu & -\mu & 0 & \mu & U_2 \\ 0 & \lambda + 3\mu & \lambda & -(\lambda+\mu) & U_4 \\ 0 & 0 & \lambda + 2\mu & -(\lambda + 2\mu) & V_2 \\ 0 & 0 & 0 & \lambda + 3\mu & V_4 \end{array}$$



$X_1 = l \qquad X_2 = 2l \qquad X_3 = 2l$

$Y_1 = 0 \qquad Y_2 = 0 \qquad Y_3 = l$

$$[K(1)] = 1/2 \begin{pmatrix} (\lambda + 2\mu) & 0 \\ 0 & \mu \end{pmatrix} \begin{matrix} U_2 \\ V_2 \end{matrix}$$

$$\Sigma = K^2$$

$$1/2 \begin{pmatrix} 2\lambda+6\mu & -2\mu & 0 & 0 \\ -2\mu & -2\lambda+6\mu & 0 & 0 \\ 0 & 0 & 2\lambda+2\mu & -2(\lambda+2\mu) \\ 0 & 0 & -2(\lambda+2\mu) & 2\lambda+6\mu \end{pmatrix} \begin{pmatrix} U_2 \\ U_4 \\ V_2 \\ V_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \rho L/2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda+3\mu & -(\lambda+2\mu) \\ -(\lambda+2\mu) & \lambda+3\mu \end{pmatrix} \begin{pmatrix} v_2 \\ v_4 \end{pmatrix} = \begin{pmatrix} Pl_2 / 2 \\ 0 \end{pmatrix}$$

Hypothèse cinématique

Segment de droite $\perp$ plan moyen ont un déplacement de corps rigide et restent $\perp$ au plan moyen

$f(x,y) \quad U_m(x,y)$

$\qquad \omega(x,y)$

$$U^* \quad \begin{matrix} u^* - zW_{,x}^* \\ v^* - zW_{,y}^* \\ x^* \end{matrix}$$

$\varepsilon^* = \varepsilon_o^* + z_\gamma^*$

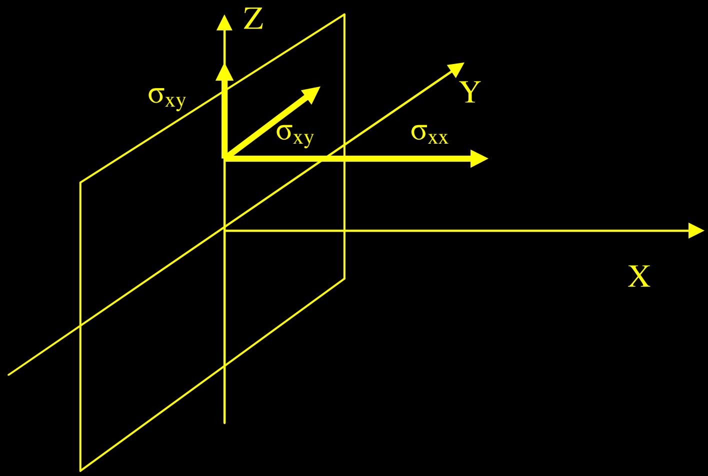
$$\varepsilon^* = \boxed{\begin{matrix} u_{,x}^* & 1/2(u_{,y}^*+v_{,x}^*) \\ 1/2(u_{,y}^*+v_{,x}^*) & v_{,y}^* \end{matrix}} + z \boxed{\begin{matrix} -W_{,xx}^* & -W_{,xy}^* \\ -W_{,xy}^* & -W_{,yy}^* \end{matrix}}$$

$P_{int}^* = -\, {}_{sm}\int \;\; N : \varepsilon_o^* \; + M : \; \gamma^* \,]\, dS$

$N : \varepsilon_o^* \;\Leftrightarrow\; Tr\;\; [N\, \varepsilon_o^*] \Leftrightarrow N_{ij}\;\; \varepsilon^*_{ij}$

où $N = {}_{(-h/2)}\int^{(h/2)} \sigma\, dz$ et $M = {}_{(-h/2)}\int^{(h/2)} z\, \sigma\, dz$

On peut séparer les problèmes de tension des problèmes de flexion



Pb de tension = Pb de contrainte planes

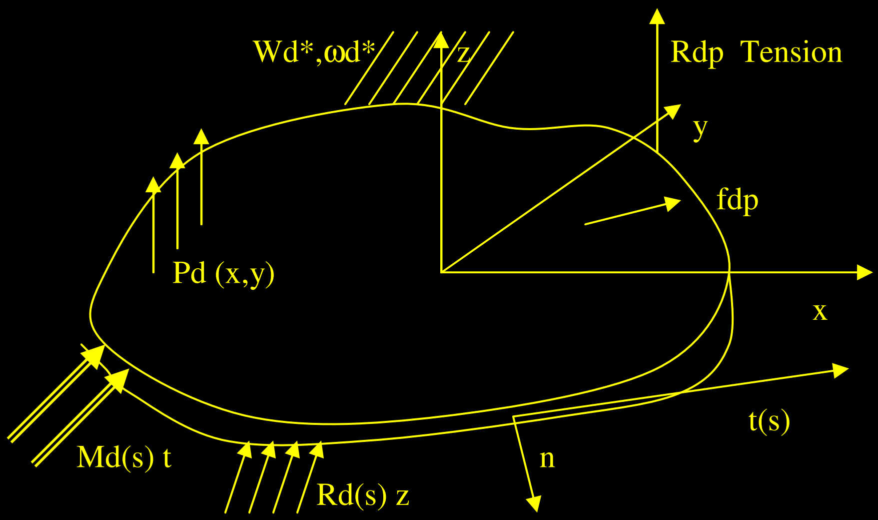
$\Rightarrow \mathrm{div}\, N \; + \; f_{\,dq} = 0$

Pb de flexion

Th PV

$P_{int}^* + P_{ext}^* = 0 \;\forall\; u^*$

$P_{int} = -\int M_{xx}\, (-W_{,xx}^*) + M_{yy}(-W_{,yy}^*) + 2\, M_{xy}\, (-W_{,xy}^*)\, dS$



$$P_{int}^* = - \int \underbrace{M_{xx} (-W_{,xx}^*)}_{\gamma_{xx}^*} + \underbrace{M_{yy} (-W_{,yy}^*)}_{\gamma_{yy}^*} + \underbrace{2 M_{xy} (-W_{,xy}^*)}_{\gamma_{xy}^*} \, dS$$

$$P_{int}^* = {}_{sm}\int \operatorname{div}(\operatorname{div} M) W^* \, ds - {}_{\partial 2sm}\int [\operatorname{div} M.n + (t\,M\,n)_{,s}] W^* \, d\Gamma - {}_{\partial 2sm}\int n.M.n\,(-W^*_{,n}) d\Gamma - [t.M\,n]_p \, W^*(p)$$

$$P_{ext}^* = {}_{sm}\int p_d \, W^* \, ds + {}_{\partial 2sm}\int R_d W^* d\Gamma + {}_{\partial 2sm}\int M_d(-W_{,n}^*) \, d\Gamma + R_{dp} \, W^*(p)$$

$\operatorname{div}(\operatorname{div} M) + P_d (x,y) = 0$ sur s_m

$\operatorname{div} \sigma + f_d = 0$ en 3D

$\operatorname{div} M\,n + (t\,M\,n)_{,s} = R_d(s)$ sur $\partial_2 s_m$

force de Kirchoff

$n\,M\,n = M_d$ sur $\partial_2 s_m$

$[t\,M\,n]_p = R_{dp}$ pour les points P

$$M - {}_{(-h/2)}\int^{(h/2)} \sigma \, dz = {}_{(-h/2)}\int^{(h/2)} \boxed{\begin{matrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{matrix}} \, dz$$

Champ de déplacement (approximation)

$$U \quad \begin{matrix} u - zW_{,x} \\ v - zW_{,y} \\ w \end{matrix}$$

$\varepsilon = \varepsilon_o + z$

Relation de comportement en contrainte plane

$\varepsilon = 1+\nu / E \ \ \sigma \ - \nu/E \ \ tr [\sigma] \ Id$ en 2D ou 3D

$\varepsilon = K_{cp}^{-1} \ \sigma$

$\sigma = K_{cp} \ \varepsilon$

$$\sigma = E/1-\nu^2 \quad \begin{matrix} \varepsilon_{xx}+\nu\varepsilon_{yy} & (1-\nu)\varepsilon_{xy} \\ (1-\nu)\varepsilon_{xy} & \varepsilon_{yy}+\nu\varepsilon_{xx} \end{matrix}$$

$N = {}_{(-h/2)}\int^{(h/2)} \sigma \, dz$

$\quad = {}_{(-h/2)}\int^{(h/2)} K_{cp} \, \varepsilon \, dz$

$\quad = {}_{(-h/2)}\int^{(h/2)} K_{cp} \, (\varepsilon_o + z\,\gamma) \, dz$

$\quad = K_{cp} \, \varepsilon_o \, h$

$N = h \, K_{cp} \, \varepsilon_o$

$M = {}_{(-h/2)}\int^{(h/2)} z \, \sigma \, dz$

$\quad = {}_{(-h/2)}\int^{(h/2)} z \ K_{cp} \, \varepsilon_o \, dz$

$\quad = {}_{(-h/2)}\int^{(h/2)} z \ K_{cp} \, (\varepsilon_o + z\,\gamma) \, dz$

$$\gamma = \begin{matrix} -W_{,xx} & -W_{,xy} \\ -W_{,xy} & -W_{,yy} \end{matrix}$$

$M = K_{cp} \, \varepsilon_o \ {}_{(-h/2)}\int^{(h/2)} z \, dz + K_{cp} \, \gamma \, {}_{(-h/2)}\int^{(h/2)} z^2 \, dz$

$\quad = h^3 \ K_{cp} \ \gamma$

$$\quad = h^3 \, E / 12(1-\nu^2) \qquad \begin{matrix} -W_{,xx} -\nu W_{,yy} & (1-\nu)(-W_{,xy}) \\ (1-\nu)(-W_{,xy}) & -W_{,yy} -\nu W_{,xx} \end{matrix}$$

Pb de flexion

Trouver (u,v,w) (x,y) et M qui verifient :

- l'équilibre : $div (div M) + \Gamma_d = 0$

- les équations de liaison : $W = W_d$ sur $\partial_1 S_m$
et $-W_{,n} = \omega_d$ sur $\partial_1 S_m$

-relation de comportement : $M = D K_{cp} \gamma$

Théorème de l'énergie

$E_d = {}_{sm}\int Tr [M \gamma] dS$

$= {}_{sm}\int M : 1/D\, Kcp^{-1} M\, dS \Rightarrow E_d (M_c)$

$= 1/2\, {}_{sm}\int D K_{cp} \gamma : \gamma\, dS \Rightarrow E_d (\gamma)$

Théorème de l'énergie potentielle : La solution minimise

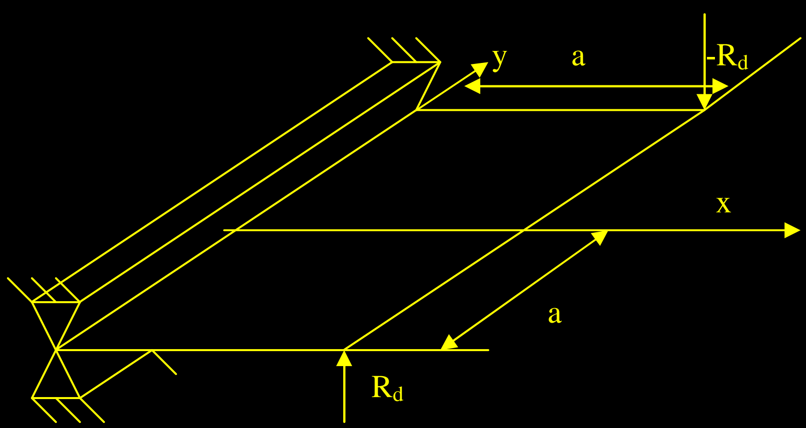
$E_p (\gamma) = E_d (\gamma) - {}_{sm}\int P_d W\, ds - {}_{\partial 2sm}\int (r_d W + M_d (-W_{,n}))\, d\Gamma - \Sigma R_{dp} W(p)$

$E_c (M) = E_d (M)$ si les déplacements imposés sont nuls

Exemple

$W_d = 0$

pour x = 0



Solution

$$M \; \boxed{\begin{matrix} 0 & M_{xy} = \alpha = cste \\ \alpha & 0 \end{matrix}}$$

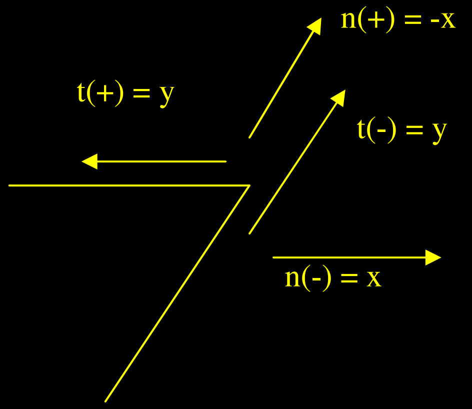
$div (div M) = 0$

équilibre OK

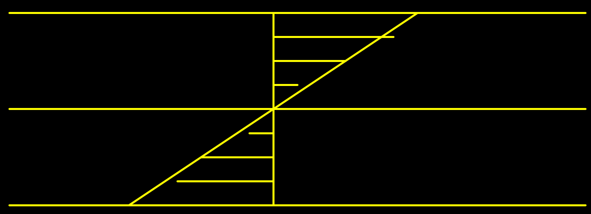
$[t . M n]_{(a,a/2)} = - Rd = (t(+)\ M n - t(-)\ M n)_{(a,a/2)} \Rightarrow \alpha = Rd/2$

$-x\ M\ y \qquad y\ M\ x$

$-\alpha \qquad m_{xy} = \alpha$



$[f(x)] = f (x+) – f(x-)$



Déplacement $M_{xy} = D (1-\nu) (- W_{, xy})$

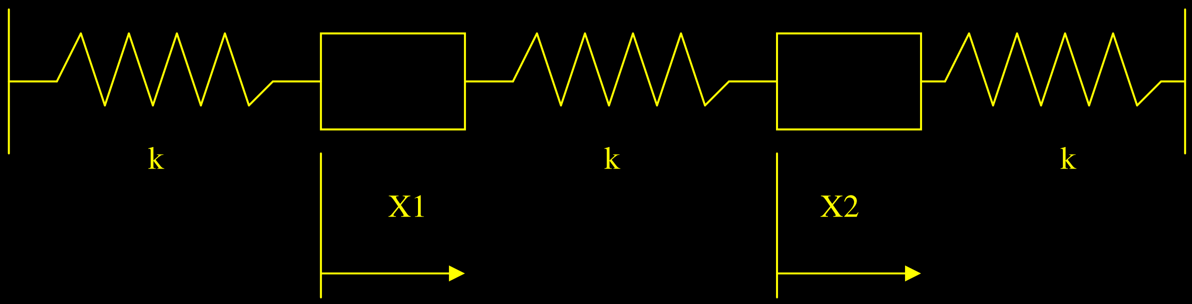
$Eh^3 / 12 (1-\nu^2) (1-\nu) (- W_{,xy}) = R_d / 2$

$W = - 6 (1 + \nu) / Eh^3 \ R_d\ xy$

Systèmes à masse et élasticité dissociées

Système à masse et élasticité associées

M E F



$M X'' + K X = F$

Système à 2 d° de liberté

pfd = sur chaque masse = $m x_1'' + 2kx_1 - kx_2 = F_1$
$m x_2'' +- kx_1 + 2kx_2 = F_2$

$$\boxed{\begin{matrix} m & 0 \\ 0 & m \end{matrix}} \begin{matrix} x_1'' \\ x_2'' \end{matrix} + \boxed{\begin{matrix} 2k & -k \\ -k & 2k \end{matrix}} \begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{matrix} F_1(t) \\ F_2(t) \end{matrix}$$

Remarque = Th de Lagrange

. Ed en 3D = 1/2 $\Omega\int$ Tr [ε (U) K ε ()] dΩ

. δ f(U) = $\partial f/U$ δU

f(x,y) : $\delta f = \partial f/\partial x \, dx + \partial f/\partial y \, dy$

$\delta E_d = \Omega \int tr [\varepsilon (U) K \varepsilon (\delta U)] \, d\Omega$
(δ (1/2 x kx) = k x δ x)
$= - T_{int}^*$ avec $u^* = \delta u$

en 1D E_d (=U) = 1/2 k x(1/2) ² + 1/2 k (X2 – X1) ² + 1/2 k X2

$T_{int}^* = - \delta E_d(= -\delta U) = -\sum_i \partial E_d/\partial x_i \, \delta x_i$ travail virtuel des efforts interieurs

$T_{int}^* = \sum_i F_i \, \delta x_i$

$T_{acc}^* = \Sigma\, m_i\, x_i''\, \delta\, x_i$
Rappel : Energie cinétique

$T = 1/2\, \Sigma\, m_i\, (x_i')^2$

$\partial T/\partial x_i' = m_i\, x_i'$

$d/dt\, (\partial T/\partial x_i') = m_i\, x_i'' \;\Rightarrow\; T_{acc}^* = \Sigma\, d/dt\, (\partial T/\partial x_i')\, \delta\, x_i$

Travail virtuel

$T_{int}^* + T_{ext}^* = T_{acc}^*$

$-\Sigma E_d/\partial x_i\, \delta x_i + \Sigma\, F_i\, \delta\, x_i = \Sigma\, d/dt\, (\partial T/\partial x_i')\, \delta\, x_i$

$-\partial E_d/\partial x_i + F_i = d/dt\, (\partial T/\partial x_i')$

$d/dt\, (\partial T/\partial x_i') + \partial E_d/\partial x_i = F_i$ Théorème de Lagrange

Ex :

$T = 1/2\, (m_1\, x_1'^2 + m_2 x_2'^2)$

$\Sigma \partial T/\partial x_i' = m_1\, x_1' + m_2\, x_2'$

$\Sigma\, d/dt\, \partial T/\partial x_i' = m_1\, x_1'' + m_2\, x_2''$

$E_d = 1/2\, kx_1^2 + 1/2k(x_2 - x_1)^2 + 1/2\, kx_2^2$

$\Sigma\, \partial E_d/\partial x_i = kx_1 + kx_2 + k(x_1 - x_2) + k(x_1 - x_2)$

$/x_1 \;\Rightarrow\; F_1 = mx_1'' + kx_1 - k(x_2 - x_1)$
$F_2 = mx_2'' + k(x_2 - x_1) + kx_2$

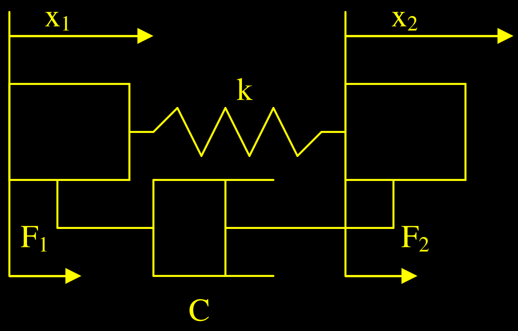
Remarque

$Ed = 1/2\, X^T\;K\;X$
$= 1/2\, [x_1, x_2] \boxed{\begin{matrix} 2k & -k \\ -k & 2k \end{matrix}} \begin{matrix} x_1 \\ x_2 \end{matrix}$

$T = 1/2\, X'^T\;M\;X'$
$= 1/2\, [\,X_1'\; X_2'\,] \begin{matrix} m & 0 \\ 0 & m \end{matrix} \begin{matrix} x_1' \\ x_2' \end{matrix}$

$D/dt(\partial T/\partial X) + \partial Ed/\partial X \;\Rightarrow\; MX'' + KX$

2.2 Système amorti



p f d = $m_1\, x_1'' \;=\; k\,(x_2 - x_1) + C(x_2' - x_1') \;+\; F_1 \;+\; \ldots$

$m_2\, x_2'' \;=\; k\,(x_1 - x_2) + C((x_1' - x_2') + F_2 + \ldots$

Ou bien en définissant un potential de dissipation

$D = 1/2\; C\; (x_2' - x_1')^2$

$E_d = 1/2\; k\; (x_2 - x_1)^2]$

Lagrange $d/dt\; (\partial T/\partial x_i') \;+ \partial E_d/\partial x_i \;+ \partial D/\partial_i \;= F_i$

$m_1\, x_1'' - k(x_2 - x_1) - C(x_2' - x_1') + \;\; = \;F_1$

=> $M\, x'' + Kx \;+ Cx' + F$

où C est la matrice d'amortissement

x mouvement libre

$M\, x'' + K\, x + Cx' \;= 0$ avec $x = X\, e^{rt}$ et $x' = rx$

$(r^2 M + K + rC\,)\; X = 0$

methode directe

$r = \pm\, i\omega$ det $(r^2 M + K + rC) = 0$

=> ω_i, Φ_i : 2d° $r^2\, m_1$ 0 possible si le nombre de d° de liberté est faible

0 $r^2\, m_1$

* même pour un mouvement forcé

$F(t) = F\sin\, \Omega_t \Rightarrow x = Xe^{i\Omega t}$

Complexe

Cas particulier où C est proportionnelle à M et K

$C = \alpha M + \beta K$

1) On résoud $Mx'' + Kx = O$

$\Rightarrow \omega_i , \Phi_i$

2) $\Phi_j^T M \Phi_i = 0$ et

$\Phi_j^T K \Phi_i = 0 \quad \forall i \neq j$

$\Phi_j^T C \Phi_i = 0$

On cherche $x = \sum_{i=1}^{n} \Phi_i q_i$

$\Phi_j^T (Mx'' + Kx + Cx' = F)$

$i=1,k \qquad \underbrace{\Phi_i^T M \Phi_i}_{\text{Masse modale } m_i} q_i'' + \underbrace{\Phi_i^T K \Phi_i}_{\text{raideur modale } k_i} q_i + \underbrace{\Phi_i^T C \Phi_i}_{\text{amortissement modal } c_i} q_i' = \Phi_i^T F$

n systèmes à 1 degré de liberté à résoudre

3. Système à n degrés de liberté

$M x'' + K x + Cx' = F$

Avec M déduit de T

$T = 1/2\, x'^T M x$

$E_d(U) = 1/2\, x^T K x$ T T V ou Lagrange

$D = 1/2\, x'^T C x'$

Dans la pratique , on fait un changement de variable

$x = A y$ avec dim (y) <<< dim x

$[x] = [A] [y]$ => Approximation

$x^T = y^T A^t$

$T = 1/2\, y'^T \underline{A^T M A} y$

M* : matrice masse réduite

$E_d = 1/2\ y^T A^T \underbrace{KA}_{K^*} y$

K* matrice de raideur réduite

Remaque :

Si A : base ds mods propres Φ

=> K* diagonale $[\Phi_1, \Phi_2, \ldots\ldots.]$

de même pour

$D = 1/2\ y'^T \underbrace{A^T C A}_{C^*} y'$

C* matrice d'amortissement réduite,

3.2 Résolution

* méthode directe (petits systèmes)

* méthode d'approximation

Changement de variables

$x = A\ y$

$[x] = [A_1, A_2, \ldots.]\ [y_1, y_2, \ldots]$ dim(y) <<<< dim(x)

Méthode de Rayleigh Ritz

Thèorème des travaux virtuels

$$\underbrace{A^T M A}_{M^*} y'' + \underbrace{A^T K A}_{K^*} y + A^T C A y' = 0$$

Lagrange : système libre

=> résolution directe

=> méthodes itératives

$y(1) = a_1\Phi_1 + a_2\Phi_2 + \ldots.$

$(K^*)^{-1}\ M^*\ y(1) = y(0) \Rightarrow \omega_1, \Phi_1$

$\omega_1, \omega_2, \omega_3$

Cas particulier dim y = 1

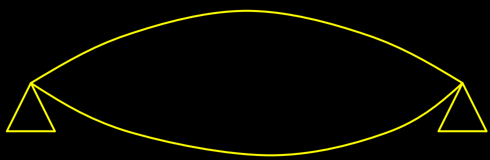
Méthode de Rayleigh

$A^T M A y'' + A^T K A y = 0$

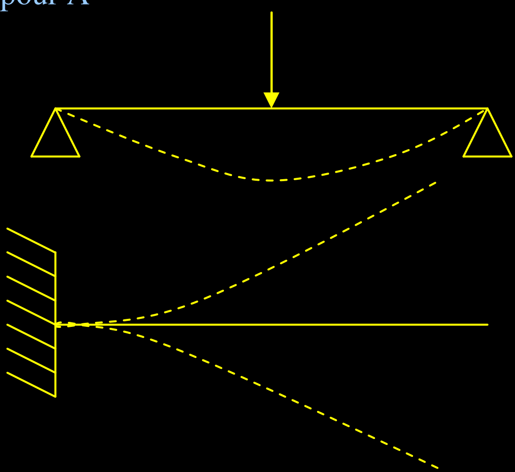
1 d° de liberté = $m^* y' + k^* y = 0$ => Solution $\omega = \sqrt{k^*/m^*}$

Ex poutre en flexion

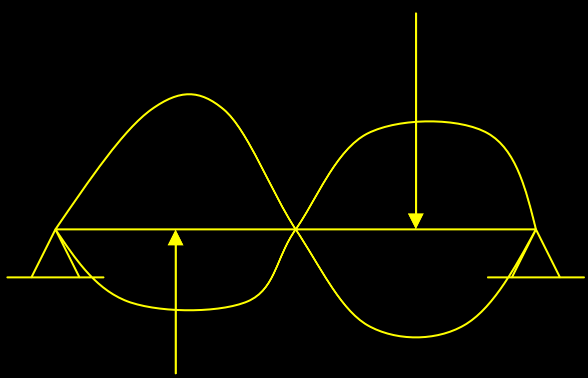
1 er mode

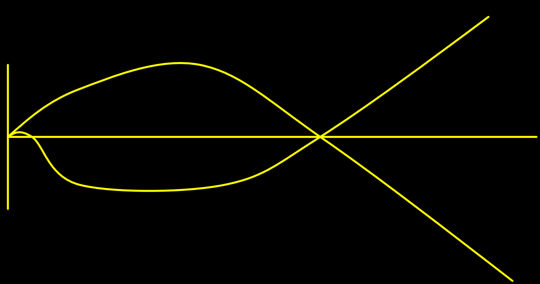


on peut prendre pour A

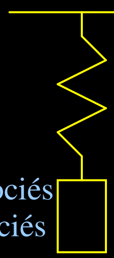


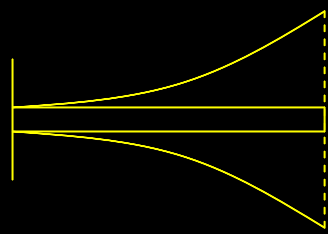
2 eme mode



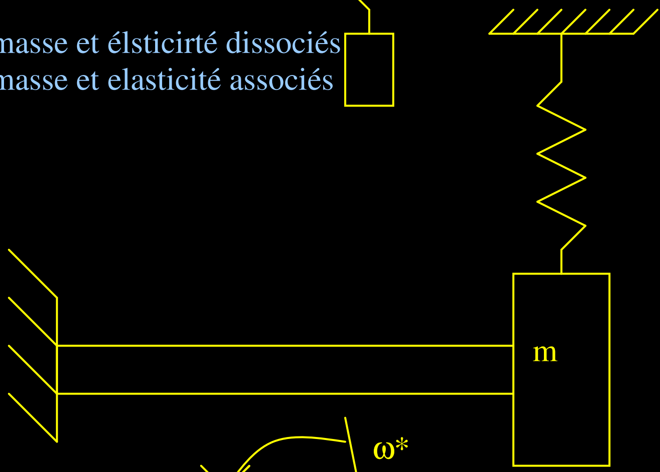


Element finis

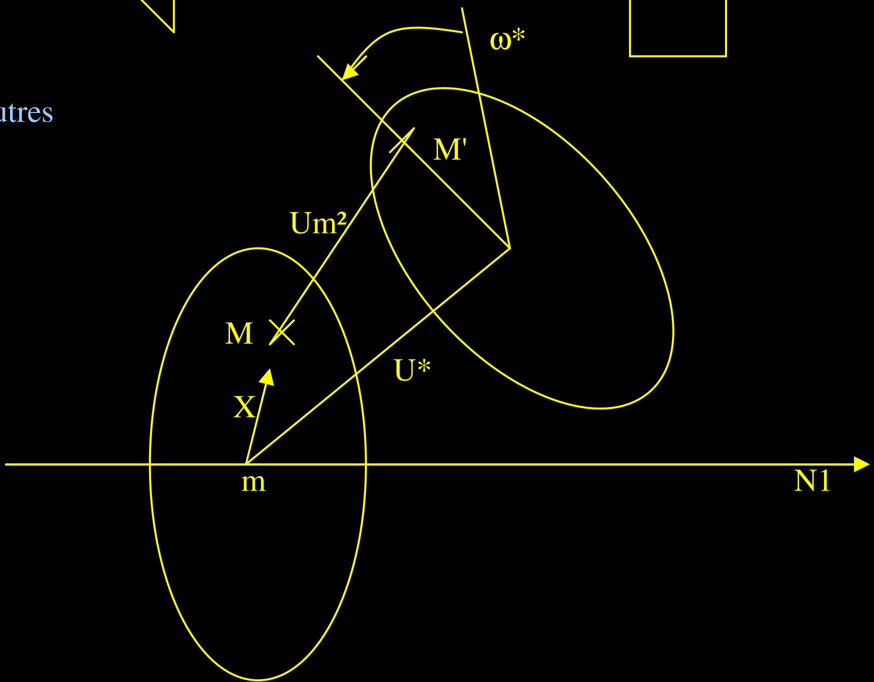




1) Système à masse et élsticirté dissociés
2) Systéme à masse et elasticité associés



1 Poutres



Théorème des travaux virtuels

$T_{int}\,^* + T_{ext}\,^* = T_{acc}^* = {}_\Omega\int \rho\, U''\, U_m'\ d\Omega$

${}_\Omega\int \operatorname{div} \sigma + f\, d = \alpha\, \gamma\ U_m^*\ d\Omega \qquad \forall\ U_m^*$

$U_m\,(x_1,x_2,x_3,t) = U_m\,(x_1,t) + \omega(x_1,t) \wedge X$

$T_{acc}^* = {}_\Omega \int \rho\ \ (u''(x_1,t) + \omega''(x_1,t) \wedge X\,)\,(\,U^* + \omega^* \wedge X\,)\, d\Omega$

remarque

${}_\Omega\int U''\ \omega^* \wedge X\, d\Omega = {}_0\int^L {}_S\int\ ''\, U_x\, \omega^* \wedge X\, d\,\Omega = 0$

$T_{acc}^* = {}_{\Omega}\int \rho\,(u''(x_1,t) + \omega''(x_1,t) \wedge X)\,(U^* + \omega^* \wedge X)\,d\Omega$

$$\begin{array}{lll} \omega_1'' \text{ torsion} & 0 & \omega_2''\, x^3 .\ \omega_3'' x_2 \\ \omega_2'' \text{ flexion} & x_2 & -\omega_1''\, x_3 \\ \omega_3'' \text{ flexion} & x_3 & \omega_1''\, x_2 \end{array}$$

$$\begin{aligned} T_{acc}^* = {}_0\int^L \rho\, u''\, u^*\, S\, dx_1 \;+\; {}_0\int^L {}_S\int [\,(\omega_2''\, \omega_2^*\, x_3^2 + \omega_3'\,'\omega_3^*\, x_2^2) \\ -(\omega_2''\omega_3^* x_2 x_3 + \omega_3''\omega_2^* x_2 x_3) \\ +\omega_1''\omega_1^*(x_2^2+x_3^2)]\,dx_1\, ds \end{aligned}$$

$$= {}_0\int^L \rho\, u''\, u^*\; dx_1 \;+\; {}_0\int^L \rho\,(\omega_2''\, \omega_2^*\, I_2 + \omega_3'\,'\omega_3^*\, I_3 + \omega_1''\omega_1^*\, I_0]dx_1$$

$$T_{int}^* + T_{ext}^* = {}_0\int^L \alpha\, U''\, U^*\, d\,x_1 + {}_0\int^L \alpha\, I\, \omega''\, \omega^*\, dx_1$$

$$I = \boxed{\begin{matrix} I_o & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{matrix}}$$

Equations locales

$$\partial R(x_1,t) / \partial x_1 + R_d = \rho\, U''$$

$$\partial M / \partial X_1 + N_1 \wedge R + M_d = \rho\, I\, \omega''$$

$$\begin{matrix} 1 & \wedge N = 0 & \\ 0 & T_2 & -T_3 \\ 0 & T_3 & T_2 \end{matrix}$$

nul si à l'origine

$$I_2 = \int x_3^{\,2}\, ds$$

$$I_3 = \int x_2^2\, ds$$

1er équation développée
PB de Traction Compression

développement de la 2° équation locale
PB de torsion

$$\left\{\begin{aligned} &\partial N/\partial x_1 + Rd_1 = \rho\, U_1''\, S \\ &\partial T_2/\partial x_1 + Rd_2 = \rho\, U_2''\, S \\ &\partial T_3/\partial x_1 + Rd_3 = \rho\, U''\, S \end{aligned}\right. \quad \Leftrightarrow \quad \left\{\begin{aligned} &\partial Mt/\partial x_1 = \rho\, I_o\, \omega_1''\, S \\ &\partial Mt/\partial x_2 - T_3 = \rho\, I_2\, \omega_2''\, S \\ &\partial Mt/x3 + T2 = \rho\, I3\, \omega3''\, S \end{aligned}\right.$$

II 1) Problème de Traction (idem en torsion)

$\partial N/\partial x_1 = \rho\ S\ U_1\ '' = \rho\ S\ \partial^2 U/\partial t^2\ (x_1,t)$

Relation de comportement

$N = ES\ \partial U_1/\partial x_1$

$\Rightarrow ES\ \partial^2 U_1/\partial x_1^2 = \rho\ S\ \partial^2 U_1/\partial t^2$

$(Mt = \mu\ I_o\ \partial\omega_1/\partial x_1 \Rightarrow \mu I_o\ \partial^2\omega_1/\ \partial x_1 = \rho\ I_o\ \partial^2\ \omega_1\ /\ \partial t^2)$

$U_1(x_1,t) = u(x)\ f(t)$

$E\ u_{,11}\ f(t) = E\ \partial^2\ U(x_1)/\partial x_1^2\ \ f(t)$
$= \rho\ U\ (x_1)\ \partial^2\ f(t)/\partial t^2$
$= \rho\ U\ (x_1)\ f\ ''\ (t)$

$E/\rho\ \ U_{,11}/U\ (x_1) = f\ ''/\ f\ (t)$
$= cste$
$= -\ \omega^2$
$\forall\ (x_1)\quad \forall\ (t)$

T.T.V. poutre en tractio,

$-\ {}_0\int^L E\ S\ \partial\ U_1\ (x,t)\ /\ \partial\ x_1\ \ U_{,1}\ *\ \ d\ x_1 = \ {}_0\int^L \rho\ S\ \ (\partial^2\ U_1\ (x_1,t))/\partial t^2\ \ U^*\ d\ x_1$

$U_{14}\ (x_1,t) = U(x_1)\ f(t)$ et $U^* = U\ (x_1)$

$-\ {}_0\int^L ES\ (\ U_{,1})^2\ dx + f(t) = {}_0\int^L \rho\ S\ u^2\ (x_1)\ d\ x_1\ f''\ (t)$

$f''/f = -\ (\ {}_0\int^L\ \ E\ S\ (\ U_{,1})^2\ dx_1)/(\ {}_0\int^L p\ S\ U^2\ dx_1) < 0$

$\Rightarrow\ \ f''\ (t) + \omega^2\ f(t) = 0$

$u_{,11} + \omega^2\ \rho\ /\ E\quad U(x_1) = 0$

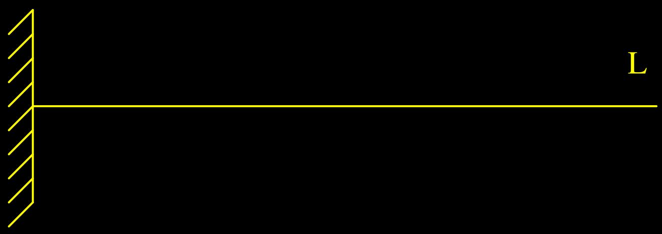
$f(t) = A\ \sin\ \omega\ (t) + B\ \cos\ \omega\ (t)$ $\qquad f'' = -\ \omega^2\ f$
pulsation propre

$u(x_1) = C\ \sin\ (\ \omega\ \sqrt{(\rho/E)}\ \ x_1\ ) + \ D\ \ \cos\ (\ \omega\ \sqrt{(\rho/E)}\ x_1)$

$\partial^2 U/\partial x_1^2\ \ = \ -\ \ \omega^2\ \ \rho\ /\ E\ U$

A, B : Condition initiale
C, D : Condition limite

Exemple



$U(o) = \quad N(L) = \; = ES\, U_{,x1}\,(L)$

$\Rightarrow\; D = 0$

$U_{,x1} = C\,\omega\,\sqrt{}\rho/E\;\; \cos\,(\,\omega\,\rho/E\;\; x_1)$

$U_{,x1}\,(L) = 0$

$= C\,\omega\,\sqrt{}\rho/E\;\; \cos\,(\,\omega\,\sqrt{}\rho/E)\;\; L)$

$\cos\,(\,\omega\,\sqrt{}\rho/E\;\; L) = 0$

$\omega_n = \sqrt{}(E/\rho L^2)\;\; (2n - 1\,)\,\Pi/2$

$n = 1\,,\,2\,,\,\ldots..$

$n=1 \qquad \omega_1 = \sqrt{}(E\Pi^2/4\rho L^2)$

$= \sqrt{}E/\rho \qquad \Pi/2L$

$\omega_2 = \sqrt{}E/\rho\;\; \Pi/2L\;\; 3$

ex :

$L = 1m$

$E = 900\ 10^6\ Pa$

$\rho = 7800\ kg\ /\ m^3$

$\omega1 = \sqrt{}900\ 10^6\ /\ 7800 \qquad \Pi/2*1$

$= 533\ rd/s \qquad f1 = 84{,}92\ Hz$

$\omega2 = 1599\ rd/s \qquad f2 = 254.76\ Hz$

Système à masse et élasticité dissociées

1 Poutres à masse et élasticité dissociées (traction = tensiont ; flexion)

Solution exacte

1.3 méthode approchées du calcul des fréquences et modes propres

Méthode de Rayleigh – Ritz => approximation sur n champs de dépracement cinématiquemebt admissible

Si n = 1 => Rayleigh
Les coefficients des fonctions déterminées par TPV ou Lagrange

$$U(x,t) = \sum_{i=1}^{n} U_i(x)\, P_i(t)$$

Energie Cinétique et déformation (poutre)

* Traction

$E_d = 1/2 \; {}_0\!\int^L N\, U_{,x}\, dx$

$\;\;\;= 1/2 \; {}_0\!\int^L ES\, U_{,x}^2 \; dx$

$\;\;\;= 1/2 \; {}_0\!\int^L ES \,[\, \partial U(x,t) / \partial x \,]^2\, dx$

$T = 1/2 \; {}_0\!\int^L \rho\, S\, V^2\, dx$

$\;\;\;= 1/2 \; {}_0\!\int^L \rho\, S \,[\, \partial U(x,t) / \partial t \,]^2\, dx$

* Torsion

$Ed = 1/2 \; {}_0\!\int^L G\, Io \,[\, \partial\omega_1(x,t) / \partial x \,]^2\, dx$

$\omega_1(x,t) = \omega(x)\, p(t)$

$T = 1/2 \; {}_0\!\int^L \rho \,[\, \partial\, \omega_1(x,t) / \partial x^2 \,]^2\, dx$

V = U2 ou U3

Methode de Rayleigh en traction

$U_1(x,t) = u\,(x)\,P(t) \qquad p' = d_p(t)/dt$

$T = 1/2\ {}_0\int^L \rho\, S\, [\, U(x)]^2\, p'^2$

$E_d = 1/2\ {}_0\int^L ES\,(\, u_{,x})^2\, dx\, p^2$

$U_{,x} = dU(x)\,/\,dx$

* Lagrange

$$\underbrace{\partial/\partial t\,[\partial T/\partial p']}_{m} \quad + \quad \underbrace{\partial Ep/\partial p}_{k} \quad = \ 0$$

$${}_0\int^L \rho\, S\,[\, u(x)]^2\, dx\, p'' \ + \ {}_0\int^L ES\,(U_{,x})^2\, dx\, p \ = \ 0$$

$$\partial/\partial t[\partial T/\partial p'] = \partial/\partial t\ 1/2\ {}_0\int^L \rho\, S\,[\, U(x)]^2\, dx\ 2\, p'$$
$$= {}_0\int^L \rho\, S\,[\, U(x)]^2\, dx\, p''$$

$$\partial\, Ep\,/\,\partial p \ = 1/2\ {}_0\int^L ES\,(U_{,x})^2\, dx\ 2\, p$$
$$= {}_0\int^L ES\,(U_{,x})^2\, dx\, p$$

$p = e^{i\omega t}$

$\omega^2 = ({}_0\int^L ES\,(U_{,x})^2\, dx\,)/({}_0\int^L \rho\, S\,(\, U\,)^2\, dx)$ 1 ère pulsation propre

* méthode de Rayleigh – ritz

$U(x,t) = u_1(x)p_1(t) + u_2\,(x)\,p_2(t) + u_3(x)\,p_3\,(t) + \ldots.$

$U(x,t) = U^T P = \underbrace{[\, U_1\,,\, U_2\,,\, U_3\,,\, ..]}_{U(x)}\ \ [\, P_1,\, P_2,\, P_3,\ldots]$

$[U(x,t)\,]^2 = U(x,t)^T\ U(x,t)$

$[u(x,t)]^2 = P^T\, U\, U^T\, P$

$T = 1/2\ {}_0\int^l \rho\, S\, \partial U(x,t)/\partial t\, dx$

$= 1/2\ p'^T\ \underbrace{{}_0\int^L \rho\, S\, U^T\, U\, dx}_{M}\ p'$

ex :

$U(x,t) = U_1(x)\, p_1(t) + U_2\,(x)\, P_2\,(t)$

$M = \ {}_0\int^L \rho\, S\, U_1^2\, dx \qquad {}_0\int^L \rho\, S\, U_1\, U_2\, dx$

$\qquad {}_0\int^L \rho\, S\, U_1 U_2\, dx \qquad {}_0\int^L \rho\, S\, U_2^2\, dx$

$ED \ = 1/2\ {}_0\int^L ES\,[\partial U(x,t)\,/\,\partial x\,]^2\, dx$

$= 1/2\ P^T \ _0\int^L ES\ U_{,x}^T\ U_{,x}\ dx\ P$

Th Lagrange

$M P'' + K P = 0$
=> système de n d° de liberté

$\partial/\partial t\ (\partial T/\partial p') + \partial Ed/\partial P = 0$

$P(t) = X e^{i\omega t}$ => $\det [\ K - \omega^2 M\ ] = 0$

1.4 Réponse à une exitation forcée

Rayleigh – Ritz T T V

$\delta\ Wext = F(t)\ [u_1(x)\ \delta\ p_1 + U_2(x)\ \delta p_2 + \ldots\ldots]$

Force généralisées

$$F = \begin{matrix} F(t)\ U_1(x) \\ F(t)\ U_2(x) \\ \vdots \\ \vdots \end{matrix}$$

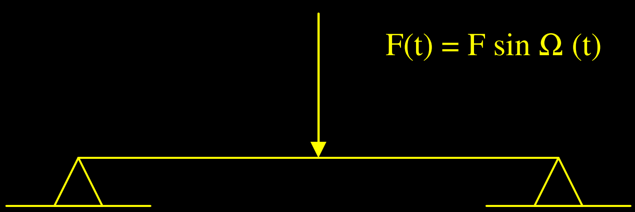
ex :

$F(t) = F \sin \Omega\ t$

$P = X \sin \Omega t$

Méthode modèle

$U(x,t) = \Phi_1\ (x)\ p_1(t) + \Phi_2(x)\ p_2(t)$ où $\Phi_i\ (x)$ mode propre du système libre



=> $\Phi_i\ (x) \sin (n\Pi/L\ \ x)$

Φ_i orthogonaux $\quad _0\int^L \rho\ S\Phi_i\ \Phi_j\ dx = 0 \qquad i \neq j$

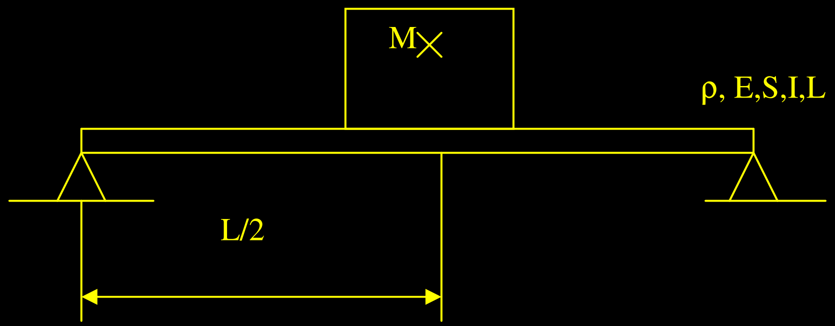
$_0\int^L ES\ \Phi_{i,x}\ \Phi_{j,x}\ dx = 0$

=> M p'' + K P = F découplé

$$\begin{bmatrix} X & & & & & & 0 \\ & X & & & & & \\ & & X & & & & \\ & & & X & & & \\ & & & & X & & \\ & & & & & X & \\ 0 & & & & & & X \end{bmatrix} p'' + \begin{bmatrix} X & & & 0 \\ & X & & \\ & & X & \\ 0 & & & X \end{bmatrix} = \begin{matrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{matrix}$$

2° Système quelconques masses et élasticités associées et dissociées

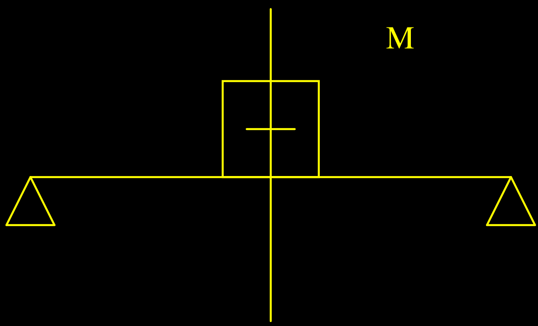
Ex

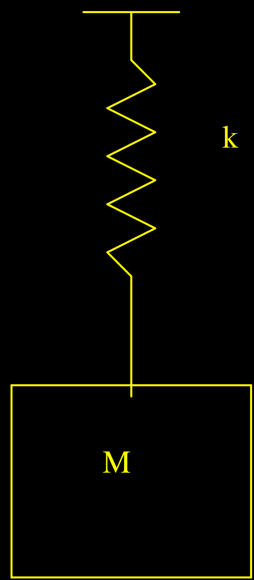


m=ρSL

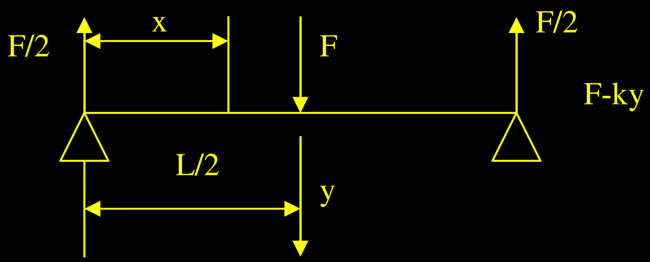
a) m <<< M

=>





k = raideur en flexion



$\omega = \sqrt{k/m}$

$Mf = F/2\ x$
$\quad = EI\ y''$

$F/2\ x^2/2 = cte = EIy'$

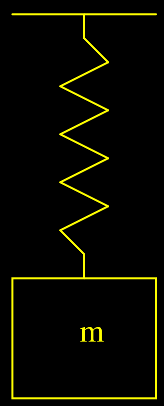
$y'\ (L/2) = 0$
$\qquad = F / 16\ L^2 + cte$

$F/4\ \ x^3/3 - F/16\ L^2\ x + cte = EIy$

$y(o) = 0 \Rightarrow cte\ = 0$

$y(x) = F/EI\ (\ x^3/12 - L^2x/16)$

$y(L/2) = F/EI\ L^3\ (1/12*8\ \ \ - -/16\ *2\ )$



$\qquad = FL^3 / 48\ EI$

$F = 48\ EI/\ L^3\ y\ (L/2)$
$\qquad k$

solution $\Rightarrow\ \omega^2 = 48\ EI / ML^3$

m non négligeable

Méthode de Rayleigh

avec la fonction

$U_2\ (x,t) = V(x)\ p(t)$

$y(x) = Fl^3 / 48\ EI\ (\ 3x/L - 4x^3/L^3)$
$U_2\ (x,t\ ) = (3x/L - 4x^3 / L^3)\ p(t)$

$V(L/2) = 1 \Rightarrow p(t)$: déplacement en (L/x)

$T, E_d \quad T = T$ poutre $+ TM$

$\omega^2 = 48\ EI / (M + 17/35\ \ m\ )L^3$

$M = 0 \Rightarrow \omega^2 / \omega exact^2$

Nom du document : DYNamique des structures.doc
Répertoire : G:\docu\etudes\cours maitrise\dynamique des structures
Modèle : E:\Documents and Settings\moi\Application Data\Microsoft\Modèles\Normal.dot
Titre : DYNamique des structures
Sujet :
Auteur : vaux
Mots clés :
Commentaires :
Date de création : 27/08/2004 13:02:00
N° de révision : 27
Dernier enregistr. le : 10/01/2006 21:45:00
Dernier enregistrement par : guiom
Temps total d'édition :2 531 Minutes
Dernière impression sur : 10/01/2006 21:46:00
Tel qu'à la dernière impression
Nombre de pages : 88
Nombre de mots : 10 380 (approx.)
Nombre de caractères : 57 093 (approx.)